The Market for Surprises: Selling Substitute Goods through Lotteries*

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Abstract

The online travel market has recently been affected by the appearance of new firms, such as hotwire.com and priceline.com, which sell hotel rooms, airplane tickets and car rentals through innovative lottery-like mechanisms. In this paper, we examine the use of lotteries over substitute goods under different market structures. We first show that a multiproduct monopolist uses lotteries over substitute goods to price discriminate among the consumers. We characterize the monopolist’s optimal selling mechanism and we show how the probability distributions of the lotteries depend on the buyers’ preferences. Then, we examine the case in which each substitute good is produced by an independent firm. Each firm is offered the possibility of selling its good also through lotteries over different firms’ goods. Lotteries are sold by third party intermediaries whenever more than one firm participates. In a market with two firms, lotteries are sold if and only if both firms are made better off. With more than two firms this is no longer true: there are equilibria in which firms sell their goods through lotteries even though they end up worse off. In this case, the sale of lotteries can shift surplus from firms to consumers.

Keywords: Price Discrimination; Optimal Selling Strategies; Lotteries

JEL classification: D42, D43, L11

1 Introduction

The recent appearance of innovative websites such as hotwire.com and priceline.com has affected the online travel market. The impact of these new players on the market is interesting because of the peculiar products they offer. Consider, for instance, how Hotwire operates in the market for hotel rooms. In addition to the standard menu of hotel rooms and respective prices, Hotwire provides the option of paying a lower price for a room in a hotel whose identity is not revealed until after the payment is made. The only information consumers have when they decide whether to accept this "blind offer" is the category of the hotel and the area where it is located. In other words, consumers can choose to pay less for an uncertain outcome. This mechanism can be described as a lottery whose prizes are the different hotel rooms that match the specified hotel

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category and location.\footnote{The mechanism used by priceline.com is more complex. The consumer selects the hotel category and the area he is interested in. Then the consumer makes a bid that may be accepted or refused by a hotel that fits the characteristics specified. Still, the mechanism can be interpreted as requiring the consumer to select a lottery and bid for it.} This innovative selling strategy has been applied successfully in the markets for hotel rooms, airline tickets\footnote{In the case of airline tickets, the flight information is revealed only after the payment. The offer specifies the number of stops and a window of 6 or 12 hours for the departure time.} and rental cars,\footnote{Priceline offers the possibility for the consumer to choose the category of the car, and an area where to pick up and leave the car. The consumer is uncertain about the rental company and the car assigned to him.} and can potentially be exported to many other markets with substitute goods.

In this paper we examine the use of lotteries over substitutes goods as a mechanism to price discriminate among the consumers under different market structures. First, we examine the use of lotteries by a monopolist that sells substitute goods. Then, we assume that each substitute good is produced by a different firm. In this case we find conditions under which third party intermediaries can sell lotteries over the different firms’ goods. Moreover, we examine the welfare impact of the introduction of lotteries in the market.

Riley and Zeckhauser (1983) were the first to examine the use of lotteries as an attempt to screen consumers. They look at a single good monopolist that attempts to price discriminate among the consumers based on the probability of delivering the good. They show that the optimal selling strategy does not include lotteries. In contrast to Riley and Zeckhauser (1983), we consider lotteries over multiple goods where the uncertainty is also about which particular good is delivered, and not only about whether any good is delivered. This type of lottery can be seen as a mechanism to screen consumers with strong preferences for a particular good from consumers that are relatively indifferent among different goods.

In this paper, we use the linear city model with a general transportation costs function to represent a market with two horizontally differentiated goods. We show that the multiproduct monopolist optimal selling strategy involves the use of lotteries (in addition to selling each good with certainty). Interestingly, consumers who buy these lotteries always get one of the goods. Furthermore, if the transportation costs function is concave, the optimal selling strategy includes just one lottery, where the consumer has equal probability of getting each good. Instead, if the transportation costs function is convex, the optimal selling strategy includes selling a continuum of type contingent lotteries. The differences in the optimal selling strategies are due to the different appeals of lotteries among consumers in the two scenarios. When the transportation costs function is concave, the consumers who value the most a given good are also the ones who value the most any lottery that awards that good with higher probability. This implies that consumers are ranked from low utility types to high utility types in the same way with respect to all lotteries that award the same good with higher probability. On the contrary, when the transportation costs function is convex, the ranking of consumers in terms of their utility from buying a lottery is different for each lottery. The consumers who have the highest utility from buying a specific good are not the ones that like the most any lottery.

We then consider an oligopoly market where each substitute good is produced by a different firm. In this context, we find conditions for competitive intermediaries to sell lotteries of the substitute goods. With two firms, this happens if and only if the presence of the intermediaries makes both firms weakly better oﬀ. The reason for this is that with only two firms, either of them can block the creation of the lottery. We define two different cases depending on whether the market is fully covered or not.

A market is fully covered if, in the absence of lotteries, all consumers buy one good in equilibrium. In this case, the introduction of lotteries makes the firms worse oﬀ and hence lotteries are not sold in equilibrium. Intuitively, firms prefer not to sell the good through lotteries
because the benefit of price discrimination between consumers is more than offset by the increase in the competitive pressure resulting from the presence of lotteries in the market. However, when the market is not fully covered, using lotteries has the added benefit of allowing firms to sell to additional consumers. We show that in this case lotteries are used in equilibrium.

With more than two firms, the results change substantially. In this case, lotteries can be sold even if firms were better off in the absence of lotteries. With more than two firms, each individual firm no longer has the power to veto the creation of a lottery, since a lottery can always be created with the substitute goods produced by other firms.

The effect of the introduction of lotteries on the social welfare depends crucially on the market coverage and on the market structure. The overall effect is determined by the market coverage: when the market is fully covered, social welfare decreases unambiguously; when the market is not fully covered, social welfare may increase. On the other end, the market structure is critical in terms of welfare distribution. Indeed, when no firm holds veto power (i.e., in the case of more than two firms), the appearance of lotteries may entail a transfer of surplus from firms to consumers.

McAfee and Mcmillan (1988) extend the result of Riley and Zeckhauser (1983) to the multi-product monopolist case using a multi-characteristics consumer model. They consider the case of two goods and, under specific conditions on the consumers’ preferences, they show that lotteries do not increase the monopolist’s profit. In a setting with two goods and two dimensional types, Thanassoulis (2004) provides some counterexamples in which the monopolist is able to increase its profits by selling lotteries. Pavlov (2011) independently reaches two of ours results for the monopolist case, namely that selling lotteries over substitute goods is optimal and that we can focus in lotteries where the consumer always gets one goof. The main difference is that Pavlov examines the case in which consumers have a utility function that is linear in their valuation for the goods, and derives the optimal mechanism for different distribution of consumers’ valuations. Instead, we consider the case in which consumers are uniformly distributed along the Hotelling line, and derive the optimal mechanism for different specifications of the consumers’ transportation costs. Our focus on different specifications of the transportation cost function allows us to point out the crucial difference between concave and convex costs. Our work is also complementary to a growing literature that analyzes the role of internet intermediaries as match-makers (Caillaud and Jullien (2003), Jullien (2001), and Rochet and Tirole (2003)). The focus of these studies are the efficiency gains provided by the appearance of such intermediaries, and their "chicken and egg" problem of persuading both sides of the market (sellers and buyers) to use it. Our angle is different. We consider an alternative rationale for the presence of intermediaries: the provision of mechanisms (i.e., lotteries) that enable competing firms to price discriminate consumers. Moreover, in our setting the appearance of intermediaries in the market may entail a decrease of the social surplus.

The paper is organized in the following way. In Section 2 we set up the basic model. The multiproduct monopolist’s optimal selling strategy is discussed in Section 3. In Section 4 we examine the possibility of lotteries being offered in markets of oligopolistic competition. Section 5 concludes the paper.

2 Model

We consider a variation of the Hotelling model of horizontal differentiation. There are two goods, indexed by $i = \{A, B\}$, located at the two endpoints of a segment $[0, 1]$. The marginal cost of producing each good is identical and, without loss of generality, is assumed to be zero. First, we assume that both goods are sold by a multiproduct monopolist. Then, we consider a duopoly in
which each good is sold by a different firm. Firm A, located at $x = 0$, sells good A and firm B, located at $x = 1$, sells good B.

There is a continuum of unit demand consumers uniformly distributed along the segment. Each consumer’s preference over the goods A and B is private information.

The utility of good A for a consumer located at $x$, where $x$ is the distance from good A, $x \in [0, 1]$, is given by

$$U_A(x) = V - c(x)$$

where $V$ is a positive constant value, and $c(x)$ is a generic transportation cost function. We assume that $c(\cdot)$ and $c'(\cdot)$ are continuous functions, $c(0) = 0$, and $c'(x) > 0$. Similarly, the utility of good B is

$$U_B(x) = V - c(1 - x)$$

If a consumer does not buy any good, his utility is zero.

3 Monopoly

In this section, we consider a monopolist that sells two horizontally differentiated goods, A and B. Given the assumptions on the distribution of consumers along the segment and on the transportation costs function $c(\cdot)$, we divide the segment $[0, 1]$ into two sub-segments, $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, and we solve the profit maximization over the second sub-segment. Indeed, the general profit maximizing solution is obtained by combining the symmetric solutions for the two sub-segments.

Applying the Direct Revelation Principle, without loss of generality, we can limit our attention to direct mechanisms, where each consumer reveals his location $x$ to a fictitious mediator who then selects an outcome for him. The set of feasible outcomes we consider includes stochastic outcomes. We define an outcome as a combination of a probability $r$ of obtaining good A, probability $q$ of obtaining good B and a corresponding price, $p(r, q)$. We refer to these probabilistic outcomes $(r, q)$ as lotteries.

The monopolist’s problem (MP) can be formalized in the following way

$$\text{(MP)} \quad \max_{p, q} \int_0^1 p(x) \, dx$$

s.t. \quad IR \quad r(x) U_A(x) + q(x) U_B(x) - p(x) \geq 0 \quad \forall x$$

$$\text{IC} \quad r(x) U_A(x) + q(x) U_B(x) - p(x) \geq r(y) U_A(x) + q(y) U_B(x) - p(y) \quad \forall y, x$$

$$q(x) \geq 0, \, r(x) \geq 0; \forall x \in [\frac{1}{2}, 1]$$

Where $p(x)$ is the price of a lottery $(r(x), q(x))$. The MP is subject to the individual rationality constraint (IR) and the incentive compatibility constraint (IC) of each consumer.

3.1 Sure Prize Lotteries

We show that, under certain conditions, in order to find the profit maximizing solution, we need to consider only lotteries $(1 - q, q)$, with $q \in [0, 1]$. These are lotteries where consumers always win one good as prize and all uncertainty is about which good is awarded. We call them Sure Prize Lotteries.
The monopolist sells only Sure Prize Lotteries when two conditions are met. First, consumers in the middle of the segment have higher utility from the least preferred good than from the outside option. Second, the utility of the least preferred good is relatively higher for types closer to the middle of the segment. When these conditions hold, shifting probabilities from the outside option to the least preferred good increases the consumer’s surplus that the monopolist can extract, and relaxes the incentive compatibility constraint. In our setting both conditions are satisfied when $V$ is high enough.

We introduce two definitions to better explain this result.

**Definition 1** Given a set of lotteries $I = \{i\}$, a lottery-mechanism $\alpha$ is a tuple $(q_i^\alpha, (r_i, r_i))_{i \in I}$ where, given each lottery $i$ and the associated probabilities $(q_i, r_i)$, the prices $p_i^\alpha$ are such that the lottery-mechanism $\alpha$ is individually rational, incentive compatible, and maximizes the monopolist’s revenue.

**Definition 2** Given a set of lotteries $I = \{i\}$, lottery-mechanism $\alpha$ dominates lottery-mechanism $\beta$ if $\alpha$ generates more revenue than $\beta$.

**Proposition 1** Given a lottery-mechanism $\alpha$, $(p_\alpha^i, (\pi_i, \chi_i))_{i \in I}$, such that, for some $i$, $\pi_i + \chi_i < 1$, there always exists a lottery-mechanism $\beta$, $(p_\beta, (1 - \chi_i, \chi_i))_{i \in I}$, that dominates $\alpha$.

**Proof.** See Appendix A. $

Proposition (1) implies that, without loss of generality, we can restrict the MP to lotteries of type $(1 - q, q)$. Define $U(x, q)$ as the expected gross utility that a consumer of type $x$ derives from lottery $q$

$$U(x, q) = V - (1 - q)c(x) - qc(1 - x)$$

Notice that the standard single crossing condition (SC) holds. This means that the marginal utility increases with the type $x$

$$\text{(SC)} \quad \frac{\partial U(x, q)}{\partial x \partial q} = c'(x) + c'(1 - x) > 0$$

Let $W(x, q(y))$ denote type $x$’s net utility of buying lottery $q(y)$

$$W(x, q(y)) = V - (1 - q(y))c(x) - q(y)c(1 - x) - p(y)$$

We can restate the MP as

$$\max_{p, q} \int_\frac{1}{2}^1 p(x)\, dx$$

s.t. \text{IR} \quad W(x, q(x)) \geq 0 \quad \forall x$$

$$\text{IC} \quad W(x, q(x)) \geq W(x, q(y)) \quad \forall y, x$$

$$0 \leq q(x) \leq 1; \forall x \in [\frac{1}{2}, 1]$$

When all types participate and the SC holds, the IC are verified if and only if the usual local optimality (LO) and monotonicity conditions hold (see Salanié (2008)).

5
Lemma 2  The IC are verified if and only if
\[
\text{(LO)} \quad \frac{dW(x)}{dx} = -(1 - q(x))c'(x) + q(x)c(1 - x)
\]
\[
\text{(Monotonicity)} \quad q(x) \text{ non-decreasing}
\]

The proof is standard and hence is omitted.

Consider now the IR constraints. In the standard screening problem, different types of consumers obtain the same utility from the lowest quality/quantity, which is usually set at zero and assumed to be the outside option. This condition, together with the SC and the monotonicity conditions, imply that if the IR of the lowest type holds, then all other IR constraints also hold. This simplifies the MP since we only need to guarantee that the IR constraint of the lowest type holds.

However, in our model, as opposed to the standard problem, the utility associated with the lowest quality, \( q = 0 \), given by \( U(x) = V - c(x) \), decreases with the type \( x \).

In appendix B, we show that if \( x \in \left[ \frac{1}{2}, 1 \right] \) the optimal solution implies \( q(x) \geq \frac{1}{2} \). Next, we show that if the LO holds and \( c(\cdot) \) is concave, then the net utility, \( w(x) \), increases with type \( x \), for \( q \geq \frac{1}{2} \).

Lemma 3

If \( c(\cdot) \) is weakly concave, then LO implies that \( \frac{dW}{dx} \geq 0 \), \( \forall x \in \left[ \frac{1}{2}, 1 \right] \) and \( q \geq \frac{1}{2} \) \ (1)

Proof. The LO is positive iff
\[
\frac{dW}{dx} \geq 0 \iff q \geq \frac{c'(x)}{c'(x) + c(1 - x)} \quad (2)
\]
If \( c(\cdot) \) is weakly concave, then \( c'(x) \leq c(1 - x) \) for \( x > \frac{1}{2} \). Therefore, \( q \geq \frac{1}{2} \) guarantees that the inequality (2) holds.

When \( c(\cdot) \) is concave, LO implies that if the IR constraint of the lowest type \( x = \frac{1}{2} \) holds, then the IR of all other types also hold. Therefore, we can solve the MP in the standard way considering only the IR of the lowest type. However, when \( c(\cdot) \) is convex, LO does not guarantee that the net utility increases with the type.

Since the method to solve the MP turns out to be different depending on the shape of the transportation costs function, we consider separately the case of concave costs and convex transportation costs.

3.2 Concave Transportation Costs

When the transportation costs are concave, the consumers located closer to the extreme of each sub-segment value all the lotteries with higher probability of obtaining their most preferred good more than consumers located closer to the middle of the segment. Therefore, the model becomes a standard model of price discrimination where consumers can be ranked from the lowest type, in the middle of the segment, to the highest type, at the extreme of the segment.

Proposition 4 If \( c(x) \) is concave, then the firm sells the two basic goods and only one lottery with probability \( q = \frac{1}{2} \). There is a \( x^* \in \left( \frac{1}{2}, 1 \right) \) such that consumers of type \( x \in [0, 1 - x^*] \) buy good A; consumers of type \( x \in (1 - x^*, x^*) \) buy the lottery, and consumers of type \( x \in [x^*, 1] \) buy
good B. The price of the lottery is \( p_L = V - c \left( \frac{1}{2} \right) \) and the prices of goods are \( p_A = p_B = V - c \left( \frac{1}{2} \right) + \frac{1}{2} (c(x^*) - c(1 - x^*)) \). Where \( x^* \) is given by

\[
x^* = 1 - \frac{c(x^*) - c(1 - x^*)}{c'(x^*) + c'(1 - x^*)}
\] (3)

**Proof.** Using lemma 2 and 3 we can restate the MP as

\[
\max_{p,q} \int_{\frac{1}{2}}^{1} p(x) \, dx
\]

s.t. (LO)

\[
\frac{\partial W(x)}{\partial x} = -(1 - q)c'(x) + qc'(1 - x)
\]

(Monotonicity) \( q(x) \) non-decreasing

(IR of \( x = \frac{1}{2} \)) \( W(\frac{1}{2}, q(\frac{1}{2})) \geq 0 \)

\( q(x) \geq \frac{1}{2}; \forall x \in [\frac{1}{2}, 1] \)

Integrating the LO we obtain

\[
W(x) = \int_{\frac{1}{2}}^{x} \left( -qc'(z) - (1 - q)c'(1 - z) \right) dz + W \left( \frac{1}{2} \right)
\] (4)

At the optimum the IR of the lowest type, \( x = \frac{1}{2} \), is binding so that \( W \left( \frac{1}{2} \right) = 0 \). We can express \( p(x) \) as

\[
p(x) = V - (1 - q(x)) c(x) - q(x) c(1 - x) - W(x)
\] (5)

Using (4) to replace \( W(x) \) in (5), we obtain

\[
p(x) = V - (1 - q) c(x) - qc(1 - x) - \int_{\frac{1}{2}}^{x} \left( -qc'(z) - q c'(1 - z) \right) dz
\]

We can reformulate the MP as

\[
\max_q \int_{\frac{1}{2}}^{1} \left\{ V - (1 - q) c(x) - qc(1 - x) + \int_{\frac{1}{2}}^{x} \left( -qc'(z) - q c'(1 - z) \right) dz \right\} \, dx
\] (6)

If we integrate by parts the last term

\[
\int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{x} \left( -qc'(z) - q c'(1 - z) \right) dz \, dx
\]

\[
= \int_{\frac{1}{2}}^{1} \left( -qc'(x) - q c'(1 - x) \right) (1 - x) \, dx
\]

Hence, expression (6) becomes

\[
\max_q \int_{\frac{1}{2}}^{1} \{ V - (1 - q) c(x) - qc(1 - x) - (1 - q) c'(x) - q c'(1 - x) \} (1 - x) \, dx
\]
The maximization of \( \pi \) with respect to the schedule \( q(x) \) requires the term under the integral to be maximized with respect to each \( q(x) \) for any \( x \). Taking the first order conditions we get

\[
-c(x) + c(1-x) + [c'(x) + c'(1-x)](1-x) \, dx \quad \forall x
\]

Expression (7) does not depend on \( q \). Hence, for each type \( x \) it is either positive or negative. If it is positive, the optimal solution is to assign to type \( x \) a lottery with the highest possible probability \( q \), \( q = 1 \). If it is negative, the solution is \( q = \frac{1}{2} \). Setting equation (7) equal to zero, we can determine the threshold value \( x^* \).

\[
x^* = 1 - \frac{c(x^*) - c(1-x^*)}{c'(x^*) + c'(1-x^*)}
\]

In the optimal mechanism, higher types with \( x \geq x^* \) obtain their most preferred good for sure (lottery with \( q = 1 \)); while types with \( \frac{1}{2} < x < x^* \) obtain a lottery with \( q = \frac{1}{2} \).

Next we show that \( x^* \) is an interior solution in the interval \( [\frac{1}{2}, 1] \). At \( x^* = \frac{1}{2} \), the left side of the equation (8) is smaller than the right side of the equation, \( \frac{1}{2} < 1 \). For \( x^* = 1 \) the left side of the equation (8) is higher than the right side (since \( c(x^*) > c(1-x^*) \)). Therefore, given that \( c \) and \( c' \) are continuous functions, by the Bolzano’s theorem, there is a solution \( x^* \) in the segment \( [\frac{1}{2}, 1] \).

We can solve in a similar way the problem for the sub-segment \( [0, \frac{1}{2}] \). Hence, the optimal mechanism is a set of three lotteries with different probabilities of getting good B, \( q = 0 \), \( q = \frac{1}{2} \) and \( q = 1 \). The price \( p \) of lottery \( q = \frac{1}{2} \) is pinned down by the IR of the lowest type, \( x = \frac{1}{2} \).

\[
p\left(\frac{1}{2}\right) = V - c\left(\frac{1}{2}\right)
\]

The price \( p_B \) (or \( p_A \)) is determined by making the IC of type \( x = x^* \) (or \( x = 1 - x^* \)) binding

\[
p_B = V - c(x^*) - \left[ V - \frac{1}{2}c(x^*) - \frac{1}{2}c(1-x^*) \right] - \left( V - c\left(\frac{1}{2}\right) \right)
\]

\[
= V - \frac{1}{2} \left[ c(x^*) - c(1-x^*) \right] - c\left(\frac{1}{2}\right)
\]

\[\Box\]

We have shown that when transportation costs are concave, the multiproduct monopolist optimal selling strategy is to sell only one lottery with probability \( \frac{1}{2} \). This result is in stark contrast with the no lottery result for a single product monopolist reported in Riley and Zeckhauser (1983).\(^4\)

The intuition is as follows. Consider first the single product monopolist case. The gross expected utility of a lottery with probability \( q < 1 \) over the utility of obtaining the good for sure is constant across all the consumers, independent of their location \( x \) (i.e. \( \frac{q(V-c(1-x))}{V-c(1-x)} = q \)). This makes price discrimination not optimal.

In contrast, in the multiproduct monopolist case, the relative expected utility of a lottery \( (1-q,q) \) compared with the utility of obtaining good B for sure, denoted by RU, depends on of the consumer’s location \( x \).

\(^4\)Notice that the "haggling" strategy may be revenue improving in a context with two goods, even when "haggling" implies the risk for the consumer to pay for nothing (i.e. buying a lottery with positive probability of not winning any good.)
The next Lemma shows that RU decreases with the type x

**Lemma 5** RU decreases with the type x

**Proof.**

\[
RU = \frac{(1 - q)(V - c(x)) + q(V - c(1 - x))}{V - c(1 - x)} = q + \frac{V - c(x)}{V - c(1 - x)}
\]

which is decreasing in \( x \). ■

Interestingly, the single lottery result with concave costs can still be seen as an extension of the no lottery result of Riley and Zeckhauser (1983) to the multiproduct case. Assuming that the value \( V \) is high enough and the monopolist wants to cover all the market, we can restate his problem in the following way. The monopolist sells a lottery with probability \( q = \frac{1}{2} \) to all consumers. Then, on top of this, he sells an additional option \( S \) of trading the lottery for the consumers' favorite good with probability \( s \), \( 0 < s < 1 \). Selling this option is formally equivalent to also sell another lottery with probability different than \( q = \frac{1}{2} \).

This problem can be formalized in the following way. Let the utility from buying a \( \left( \frac{1}{2}, \frac{1}{2} \right) \) lottery be the outside option

\[
\frac{1}{2} (V - c(x)) + \frac{1}{2} (V - c(1 - x))
\]

and redefine both the utility of buying the good and the utility of the option \( S \) as the increase in utility compared with the outside option.

The utility of buying good \( B \) is

\[
\hat{U}_B(x) = [V - c(1 - x)] - \left[ \frac{1}{2} (V - c(x)) + \frac{1}{2} (V - c(1 - x)) \right] = \frac{1}{2} c(x) - \frac{1}{2} c(1 - x)
\]

The utility of the option \( S \) is

\[
\hat{U}(x, s) = s (V - c(1 - x)) + (1 - s) \left[ \frac{1}{2} (V - c(x)) + \frac{1}{2} (V - c(1 - x)) \right] - \left[ \frac{1}{2} (V - c(x)) + \frac{1}{2} (V - c(1 - x)) \right] = q \left( \frac{1}{2} c(x) - \frac{1}{2} c(1 - x) \right)
\]

Notice that in each half of the segment line, the demand for option \( S \) can be seen as the demand of a single good monopolist. Assuming concave transportation costs, consumers located at the extreme of the segment have the highest willingness to pay for the option \( S \), while the consumer located at \( x = \frac{1}{2} \) is willing to pay zero.

However, the relative utility of buying an option \( S \) compared with the utility of buying good B for sure is constant across types

\[
\frac{\hat{U}(x, s)}{\hat{U}_B(x)} = \frac{s \left( \frac{1}{2} c(1 - x) - \frac{1}{2} c(x) \right)}{\frac{1}{2} c(1 - x) - \frac{1}{2} c(x)} = s
\]

Here, we can apply the Riley and Zeckhauser [7] result to show that the monopolist does not offer lotteries of this extra option. Hence, the optimal selling strategy is to offer only one lottery with probability \( q = \frac{1}{2} \) and selling both goods for sure.
3.3 Convex Costs

When the transportation costs are convex it is no longer true that consumers at the extreme of the segments value lotteries more than consumers closer to the middle of the segment. Hence, we can no longer classify consumers as high value or low value consumers. The LO no longer implies that $w(x)$ is increasing. Therefore, we can no longer apply the Mirrlees’ [6] integration by parts technique.

The following proposition describes the monopolist’s optimal mechanism when the transportation costs are convex.

**Proposition 6** If the transportation costs function $c(x)$ is convex, then there is an $x^* \in \left[ \frac{1}{2}, 1 \right]$ such that consumers of type $x \in \left[ 0, 1-x^* \right]$ buy good A, consumers of type $x \in (1-x^*, x^*)$ buy a continuum of type contingent lotteries, and consumers of type $x \in [x^*, 1]$ buy good B. Goods A and B are sold at prices $p_A = p_B = V - c(x^*)$; a continuum of lotteries with probabilities $1-q = \frac{c'(1-x^*)}{c'(1-x)+c'(x)}$ for any $x \in \left[ \frac{1}{2}, x^* \right]$ are sold at prices $p(q) = V - q + q^2$, and a continuum of lotteries with probabilities $1-q = \frac{c'(x)}{c'(1-x)+c'(x)}$ for any $x \in [1-x^*, \frac{1}{2}]$ are sold at prices $p(q) = V + q + q^2$, where $x^*$ is given by the following condition

$$1-x^* = \frac{c(x^*) - c(1-x^*)}{c'(x^*) + c'(1-x^*)}$$

**Proof.** We show in Appendix C that there is an intermediate region where both IR and IC constraints bind. From the local IC constraints we obtain

$$p'(x) = -q'(x)(c(x) - c(1-x))$$

The IR constraints are given by

$$p(x) = V - [1-q(x)]c(x) - q(x)c(1-x)$$

Both conditions determine a sequence of lotteries and corresponding prices

$$1-q(x) = \frac{c'(1-x)}{c'(1-x)+c'(x)}$$

$$p(x) = V - \frac{c'(1-x)}{c'(1-x)+c'(x)}c(x) - \frac{c'(x)}{c'(1-x)+c'(x)}c(1-x)$$  \hspace{1cm} (9)$$

4 Competition

In this section, we consider the case of a duopoly: each good, A and B, is sold by a different firm. By $i = \{A, B\}$ we will refer interchangeably to the goods or to the firms. We assume that transportation costs are weakly concave. In addition to selling their goods separately, firms can also sell them as prizes of lotteries sold by intermediaries.\(^5\) For simplicity, we consider a lottery in which consumers obtain each good with probability $\frac{1}{2}$, denoted by $\left(\frac{1}{2}, \frac{1}{2}\right)$ lottery. We focus on a $\left(\frac{1}{2}, \frac{1}{2}\right)$ lottery because it is the one used by a two-goods monopolist to optimize its profits when $c(.)$ is weakly concave. We assume that the intermediaries have no cost of organizing

\(^5\)The lotteries could also be organized by the two firms.
lotteries and operate in a perfectly competitive market. An alternative assumption would be to
consider an environment with just one intermediary with market power. However, we want to
focus on the impact of competition on the ability of firms to use lotteries and compared it with
the monopolist case. The presence of an additional strategic player with market power would
have added one extra factor that could potentially affect the optimallity of the use of lotteries.
Furthermore, it will increase the complexity of the analysis.

We assume that each firm \( i \) receives a fraction of the lottery revenues equal to the proportion
of lottery buyers that receive good \( i \). Therefore, the expected lottery revenue of each firm is \( \frac{1}{2} \)
of the total lottery revenue.

Notice that consumers only buy lotteries if its price is lower than the price of the goods,
otherwise each customer would prefer to buy his favorite good. Hence, intermediaries cannot
simply buy the goods at the same price as customers and resell them through lotteries. Firms
must be willing to sell the goods through the lottery at a price that is lower than the price of
the goods. Notice also that consumers only consider buying the lowest price lottery ticket within
the set of lotteries offered in the market.

The timing of the game is as follows. At time \( t = 1 \), each firm declares the set of prices at
which it is willing to sell its good through the lottery. Lotteries are offered in the market only
at prices accepted by both firms. At time \( t = 2 \), after learning which lotteries are offered in the
market, firms simultaneously choose the prices at which to sell their respective goods separately.
Finally, at time \( t = 3 \) consumers make the purchasing decisions.

We consider two distinct cases. First, we examine a market where, in the absence of lotteries,
all consumers buy a good in equilibrium. We say that this market is fully covered. Then, we
consider a market where, in the absence of lotteries, some consumers close to the middle-point
of the segment do not buy any good in equilibrium.

**Definition 3** A market is fully covered if and only if, in the absence of lotteries, all consumers
buy a good in equilibrium.

The distinction between fully covered and not fully covered markets is important both in
terms of the equilibria that arise and the welfare impact. When the market is fully covered, the
following proposition holds.

**Proposition 7** If the market is fully covered, the social welfare in the equilibrium with lotteries
is lower than the social welfare in the symmetric equilibrium when lotteries are not allowed.\(^6\)

**Proof.** If the market is fully covered, the welfare is maximized in the symmetric equilibria without
lotteries. This is because every consumer buys his preferred good. With lotteries, consumers do
not always end up with their favorite good. This implies higher transportation costs and a lower
social welfare. Hence, the social welfare is lower than in the absence of lotteries. \( \blacksquare \)

Notice that this result is very general. In particular, it holds regardless of the market struc-
ture.

The following proposition determines when the market is fully covered.

**Proposition 8** There is a threshold value \( V^* \) such that if and only if \( V \geq V^* \) the market is
fully covered. The value \( V^* \) is derived from the following system of equations

\[
V^* = c \left( \frac{1}{2} \right) + p^v
\]

\(^6\)This result holds regardless of the market structure.
\[ p^v = \frac{c^{-1}(V^* - p^v)}{c^{-1}V((V^* - p^v))} \]

**Proof.** See Appendix D. ■

### 4.1 Fully Covered Market

In this section we assume that \( V \geq V^* \), which implies that the market is fully covered. First, we consider a market where lotteries are not allowed. Following Hotelling [4], we obtain the next proposition.

**Proposition 9** In the symmetric equilibrium without lotteries the firms’ optimal prices are given by

\[ p_{NL}^i = \frac{g^{-1}(0)}{g^{-1}(0)} \quad i = A, B \tag{10} \]

where \( g(x) = c(x) - c(1 - x) \).

The equilibrium price, \( p_{NL} \), represents an important benchmark. Indeed, we will see that in a setting with a fully covered market, lotteries would be used only if they allow each firm to charge higher prices than \( p_{NL} \) to some consumers.

Consider now the possibility of intermediaries selling firms’ goods through a lottery at a price \( p_L \). We show that under certain conditions the duopolist firms do not use lotteries even in situations where a multiproduct monopolist would use them.

**Proposition 10** If the price of the lottery and the price of each good are strategic complements, lotteries are not offered in equilibrium. Furthermore, prices are strategic complements if and only if the following condition holds

\[ \frac{[g^{-1}(\cdot)] [g^{-1}(\cdot)]}{[g^{-1}(\cdot)]^2} \leq \frac{3}{2} \tag{11} \]

or

\[ \frac{[g^{-1}(\cdot)] [g^{-1}(\cdot)]}{[g^{-1}(\cdot)]^2} \geq 2 \]

Condition (11) is verified by most well behaved concave transportation cost functions. For example, the linear and square root functions satisfy this condition.\(^7\)

**Proof.** Let \( p_L \) denote the minimal price of the lottery accepted by both firms at time \( t = 1 \). At time \( t = 3 \), the consumer indifferent between good \( A \) and the lottery, located at \( x \), is given by

\[ V - c(x) - p_A = V - \frac{1}{2}(c(x) + c(1 - x)) - p_L \]

which simplifies to

\[ g(x) = 2(p_L - p_A) \]

\(^7\)Intuitively, this condition imposes that the relative value of \( g^{-1}(\cdot) \) cannot be too high. This avoids a situation where the demand of the firm becomes very inelastic as the demand decreases. This guarantees that the firm does not want to increase its price when its demand falls in response to a decrease in the lottery price.
Given that firm B’s problem is symmetric, the demand for each good is given by

\[
D_i = \begin{cases} 
0 & \text{if } g^{-1}(2(p_L - p_i)) \leq 0 \\
g^{-1}(2(p_L - p_i)) & \text{if } g^{-1}(2(p_L - p_i)) \in (0, 1) \\
1 & \text{if } g^{-1}(2(p_L - p_i)) \geq 0 
\end{cases}
\]  

(12)

At time \( t = 2 \), firm \( i \) chooses its price, \( p_i \), in order to maximize its profits for a given \( p_L \).

\[
\max_{p_i} \pi_i = p_i g^{-1}(2(p_L - p_i)) + \frac{p_L}{2} \left[ 1 - g^{-1}(2(p_L - p_i)) - g^{-1}(2(p_L - p_j)) \right] 
\]  

(13)

where \( j \neq i \) and \( j, i \in \{A, B\} \)

Firm \( i \)’s optimal price, \( p_i = h(p_L) \), is

\[
p_i = \frac{g^{-1}(2(p_L - p_i))}{2g^{-1}(2(p_L - p_i))} + \frac{p_L}{2}, \quad \text{if } p_L = p_i^{NL} 
\]  

(14)

We show in Appendix E that if condition (11) holds, the price of the lottery and the prices of the goods are strategic complements, \( h(p_L) = 0 \), and that \( h(p_L) < 1 \). There are three hypothetical cases for the price of the lottery to consider

i) \( p_L = p_i^{NL} \). Substituting \( p_L \) in equation (15) we obtain

\[
p_i = p_L \quad \text{if } p_L = p_i^{NL} 
\]  

(16)

Condition (16) implies that the lottery has no demand.

ii) \( p_L > p_i^{NL} \). Condition (16) and \( h(p_L) < 1 \) imply that if \( p_L > p_i^{NL} \) then \( p_i < p_L \). Hence, the lottery has no demand.

iii) \( p_L < p_i^{NL} \). \( h(p_L) > 0 \) and condition (16) imply that if \( p_L < p_i^{NL} \) then \( p_i < p_i^{NL} \).

Hence, the industry profits and each firm’s profit would be lower than the profit in the absence of lotteries. Therefore, at time \( t = 1 \), firms refuse to sell through the lottery at a price \( p_L < p_i^{NL} \).

We showed that under certain conditions, duopolist firms do not take advantage of the lottery to better price discriminate among consumers, charging high value consumers a high price and letting low value consumers buy the lottery at a low price. To verify the rationale behind this result, let us look at how the sale of a lottery at different price levels, \( p_L \), affects firms’ choice of the prices of the goods and their final decision of joining the lottery.

Consider first the case where the price of the lottery is lower than \( p_i^{NL} \). Condition (11) implies that the price of the goods and the price of the lottery are strategic complements. Therefore, firms would optimally set the prices of the goods at a lower level than \( p_i^{NL} \) as well. Hence, the overall industry profits and each firm’s profit would be lower than in an equilibrium without lotteries. For this reason, at time \( t = 1 \), each firm would prefer to deviate and block the appearance of the lottery at this price.

When the lottery is offered at a price higher or equal to \( p_i^{NL} \) it has two effects on the firms’ behavior that offset each other when compared with a market with no lotteries. On one hand, half of the lottery’s revenue belongs to each firm: this decreases by half the cost associated with losing each consumer when firms increase their price. Ergo, it provides an incentive for firms to set a price \( p_i \) higher than \( p^{NL} \). On the other hand, a lottery is a closer substitute of good \( i \) than is good \( j \). This makes firms’ demand twice as elastic as in a market without lottery. Hence, when the firm increases its price, it loses twice as many consumers as without lottery. Since this second
effect perfectly offsets the first effect, firms’ prices when a lottery is sold at price $p_L = p_i^{NL}$ are the same as in the a market without lotteries. Therefore, the lottery would have zero demand in equilibrium. For the same reason, when the price of the lottery is higher than $p^{NL}$ the lottery also does not have demand in equilibrium.

### 4.2 Market Not Fully Covered

In this section, we examine the introduction of lotteries in a market that is not fully covered, ergo $V < V^*$. We show that intermediaries may sell lotteries over substitute goods in equilibrium. The next proposition examines the equilibrium in the absence of lotteries.

**Proposition 11** In the symmetric equilibrium without lotteries each firm sets its price equal to

$$p_i(V) = \frac{c^{-1}(V - p_i)}{c^{-1r}((V - p_i))}, \quad i = \{A, B\}$$

and firm $i$’s profit is given by

$$\pi_i^d = p_i \left( c^{-1} (V - p_i) \right)$$

**Proof.** See Appendix D. ■

Notice that the market is not fully covered because consumers located close to the middle of the segment do not buy any good in equilibrium.

Consider now the case where competitive intermediaries sell lotteries to the consumers at no extra cost for the firms. To simplify the problem, we focus on lotteries for which the price is low enough such that all consumers derive a weakly positive surplus from buying the lottery.

**Proposition 12** When lotteries are supplied by perfectly competitive intermediaries, there are two types of subgame perfect equilibrium (SPE):

a) SPE where lotteries are not sold.
   - At time $t=1$, intermediaries offer lotteries at any price $p_L$, and each firm $i$ decides not to sell its good through a lottery at any price $p_L$
   - At time $t=2$, each firm $i$ chooses the price

$$p_i = \frac{c^{-1}(V - p_i)}{c^{-1r}((V - p_i))}, \quad i = \{A, B\}$$

b) SPE where lotteries are sold.
   - At time $t = 1$, each firm $i$ accepts to sell through a lottery of price $p_L$ if and only if $p_L = p^*$ where $p^*$ is any price that satisfies the following conditions

$$p^* \leq V - c\left(\frac{1}{2}\right)$$

$$p^* \left[ 1 - g^{-1}(2(p^* - p_i)) - g^{-1}(2(p^* - p_j)) \right] > 2p_i \left[ c^{-1}(V - p_i) - g^{-1}(2(p^* - p_i)) \right] \forall i, j$$

   - At time $t = 2$, each firm $i$ sets its price equal to

$$p_i = \frac{g^{-1}(2(p_L - p_i))}{2g^{-1}(2(p_L - p_i))} + \frac{p_L}{2}, \quad i = \{A, B\}$$

14
Proof. The proof of part a) is obvious. In the SPE with no lotteries the outcome is the same as in the market where lotteries are not allowed. No firm has an incentive to deviate and join a lottery at any price \( p_L \), since lotteries are sold only if both firms agree to sell through the lottery. The proof of part b) of the proposition is in the appendix D. □

The SPE equilibrium with no lotteries exists for any shape of the transportation costs function. The existence of the SPE equilibrium with lotteries depends on the value \( V \) and on the specification of the transportation costs function.

In the SPE where lotteries over two goods are sold, price discrimination across consumers arise. Indeed, in this equilibrium consumers without a strong preference for any good, ergo located close to middle of the segment, buy the lottery at a lower price. Consumers with a strong preference for a good buy that good at a higher price. As mentioned before, the existence of an equilibrium with lotteries depends on the gross surplus of buying the good, \( V \), and on the shape of transportation costs function. If the gross surplus of buying the good is relatively low, firms will not be interested in selling lotteries to cover the all market. Notice that the equilibrium with lotteries exists for different lottery prices. However, these prices cannot be too low, because otherwise firms’ profits would be lower than in the absence of lotteries.

In conclusion, in an uncovered market, the introduction of lotteries may increase the number of consumers who purchase the good and the industry profits.

We next show that, if the duopoly market is not fully covered, the introduction of lotteries can increase the social welfare.

Proposition 13 If the oligopoly market is not fully covered, the introduction of lotteries over substitute goods: i) might increase or decrease social welfare, ii) weakly increase firms’ profits and iii) might increase or decrease the consumers’ surplus

Proof. See Appendix F. □

5 Market with Three Firms

In a market with more than two competing firms the conditions for equilibria with lotteries to arise change significantly. In this case no firm holds a veto power on the creation of a lottery. If a firm does not sell its good through a lottery, it might still face a market where a lottery over the other two firms’ goods is sold. In this section, we show that firms may use lotteries even though their profit is lower than when lotteries are not allowed. In this case, the existence of lotteries may entail a transfer of surplus from firms to consumers.

Modeling an environment with more than two firms presents some challenges in terms of generality, tractability, and existence of equilibria.

The Salop circular city model would be the natural candidate to consider a setting with more than two firms. Consumers are uniformly distributed along a circumference of length one and firms are located on the circumference at equal distance from each other. However, in this model consumers can be indifferent between at most two goods. Moreover, in a setting with three goods, the consumers who are indifferent between two goods are those who have the lowest utility of consuming the third good. This is a key limitation of the Salop model because it makes lotteries among more than two goods unattractive. Besides, in many real world markets there are consumers who are indifferent among more than two substitute goods. Random utility models can capture these situation. However, these models become intractable analytically when we introduce lotteries.
A star-shaped model would be an alternative to have a framework that is tractable and encompasses situations of indifference among more than two goods. In this model, a set of spokes radiates from a central hub, where all the spokes are connected. The end-point of each spoke represents the location of a good. In this setting, consumers who are located closer to the central hub are relatively indifferent among all the goods. However, there is no equilibria in this model in the absence of lotteries.

In order to accommodate all these issues (i.e. generality, tractability, existence of equilibria) we propose a model that is a hybrid between the Salop Model and the Star Model: we call it the "Modified Salop" (MS) model.

For simplicity, we consider three goods, and linear transportation costs. Consumers are uniformly distributed along a circumference of length one and the three goods, A, B, and C, are located along the circumference at equal distance from each other. Consider a consumer located on the segment $ij$ between good $i$ and good $j$. Let $x$ be the distance of this consumer to good $i$. As in Salop's model, the consumer's utility of consuming $i$ and $j$ is respectively $V = \frac{1}{6} x - p_i$ and $V = \left[ \frac{1}{6} + \left( \frac{1}{6} - x \right) \right] \tau - p_j$. However, we change the utility of consuming good $z$, to the following expression

$$V = \frac{1}{6} + k \frac{1}{6} - x + h \tau - p_z, \quad 0 \leq x \leq \frac{1}{3}$$

where $k$ and $h$ are constants, $k \geq 1$ and $0 \leq h \leq \frac{1}{3}$. For simplicity we assume that $\tau = 1$.

The conditions on $k$ and $h$ guarantee that good $z$ is the least preferred for the consumers located along the segment $ij$. Furthermore, given that $k$ is positive, consumers without a strong preference for either good $i$ or good $j$ ($x$ close to $\frac{1}{6}$) are also the consumers who have the highest preference for good $z$. Notice that if $k = 1$ and $h = 0$, then a consumer at the mid-point of segment $ij$, $x = 1/6$, would be indifferent among the three goods. The role of coefficient $h$ is to guarantee the existence of equilibria in the absence of lotteries: $h$ has to be high enough so that consumers who are indifferent between $i$ and $j$ do not like too much good $z$. At the same time, if $h$ is too high, than good $z$ is never attractive to consumers located in $ij$, and the MS model is trivially equivalent to the Salop model.

**Proposition 14** In the absence of lotteries, the pure strategies equilibrium in the Salop and the MS model are the same iff $k > \frac{3h^2 + 1 + 12h}{6h} \quad$ or $\quad h \geq 0.0893$.

**Proof.** See Appendix G. ■

The difference between the Salop and the MS model is on the utility specification of the of the least preferred good. However, in the neighborhood of a symmetric equilibrium, we consider only the utilities of the two most preferred goods, which are the same in both models, to compute firms' demand curve. Hence, in a symmetric equilibrium, prices are the same in both models.

Next we examine the MS model in a market where competitive intermediaries can sell lotteries over the substitute goods. A lottery is offered in the market if two or more firms agree on the lottery price and provide their goods as prizes. For simplicity, we assume that, as in section 4, lotteries with equal probability of obtaining each good and we rule out more complex types of contracts. In particular, we assume that firms cannot be excluded from joining any particular lottery.

The timing of the game is as follows. At time $t = 1$, each firm declares the set of prices at which it is willing to sell its good through the lottery. At time $t = 2$, after learning which lotteries are offered in the market, firms simultaneously choose the prices at which they sell their respective goods. Finally, at time $t = 3$ consumers make the purchasing decisions.
Proposition 15 When competitive intermediaries are allowed to sell lotteries over the substitute goods, there are two types of subgame perfect equilibrium (SPE):

a) SPE where lotteries are not sold

b) SPE where lotteries over three goods are sold

Proof. See Appendix H. ■

Notice that we could potentially have lotteries between just two of the three goods. However, it can be shown that in our setting there is no equilibrium where lotteries over two goods are sold. We leave the complete description of the results and the formal proofs to the appendix.

Depending on the values of $h$ and $k$, different equilibria arise. Notice that we may have multiple-equilibria. Besides, different values of $h$ and $k$ sustain equilibria of type a) and type b). If $h$ or $k$ are high enough, equilibrium of type a) is the unique equilibrium of the model. In this SPE, lotteries are not sold since none of the firms are willing to sell the good through the lottery. In the second type of SPE, the three firms sell through the lottery. Interestingly, firms may be worse off in this equilibrium than in the case where no firms join the lottery.

Consider first the equilibrium with no lotteries. The simplest case is when no firm agrees to join the lottery. In this equilibrium the outcome is exactly the same as in the market where lotteries are not allowed. No firm has an incentive to deviate and join a lottery at any price $p_L$, since lottery tickets are only sold if two or more firms have agreed to sell through the lottery. Hence, this equilibrium occurs for the same values of $h$ and $k$ for which there is an equilibrium in the absence of lotteries.

Consider now the SPE with a lottery over the three goods. In this lottery consumers have a $1/3$ probability of obtaining each good. Assume, for simplicity, that firms’ strategies are such that they accept to provide their goods as prizes at the same lottery prices. At time $t = 3$, each consumer has to decide whether to buy the lottery ticket or instead to buy a good directly from its seller. Notice that consumers only consider to buy the lottery with the lowest price, $p_L$, among the lottery prices accepted by the three firms.

A consumer located in the segment $ij$ has the following expected utility from buying a lottery ticket with price $p_L$:

$$V = \frac{1}{3} x - \frac{1}{3} \left( \frac{1}{6} + \frac{1}{6} - x \right) - \frac{1}{3} \left( \frac{1}{6} + h + k \left| \frac{1}{6} - x \right| \right) - p_L , \quad 0 < x < \frac{1}{3}$$

We show in the appendix H that the demand for each good is given by

$$D_i(p_i, p_L) = \frac{1}{3} + \frac{6}{k+3} (p_L - p_i) + \frac{2h}{k+3} \quad \text{if } p_i > p_L + \frac{h}{3}, \quad i = \{A, B, C\}$$

The total demand for the lottery, which is equally divided among the three firms, is given by

$$D_L(p_A, p_B, p_C, p_L) = \frac{2}{k+3} (p_A + p_B + p_C - 3p_L - h) \quad \text{if } p_i > p_L + \frac{h}{3}, \quad i = \{A, B, C\}$$

At time $t = 2$ each firm simultaneously chooses its optimal price after learning that all firms have decided to sell the good through the lottery at price $p_L$. Firm A’s problem is given by

$$\max_{p_i} p_i D_i(p_i, p_L) + \frac{1}{3} p_L D_L(p_i, p_j, p_v, p_L) \quad i, j, v = \{A, B, C\}, \quad i \neq j \neq v \quad (21)$$
We show in Appendix H that firm i’s optimal price is given by

\[ p_i = \frac{1}{6} (k + 3) \left( \frac{h}{k + 3} + \frac{4p_L}{k + 3} + \frac{1}{6} \right) \quad i = \{A, B, C\} \] (22)

It is shown in the appendix H that the equilibrium requires that the lottery price, \( p_L \), is neither too high nor too low. If \( p_L \) is too low, joining the lottery is not profitable. If \( p_L \) is too high, the lottery would not be attractive to consumers and would have no demand in equilibrium.

For simplicity, proposition 15 provides an equilibrium with lotteries where each firm strategy consists in choosing just one lottery price, \( p_L = p^* \). It can be argued that the more natural case would be for firms to choose intervals of lottery prices. In particular firms could choose the minimal price at which they are willing to sell through the lottery. Since this case is not easily tractable, we do not fully characterize the equilibrium but we provide an example of a SPE with lotteries in appendix I for particular values of parameters of \( k \) and \( h \).

We have already shown in proposition 8 that lotteries decrease the overall surplus, as long as the market is fully covered. The next proposition addresses instead the distribution of the welfare.

**Proposition 16** If the market is fully covered, the introduction of lotteries over substitute goods in an oligopoly market: i) may increase or decrease firms’ profits. ii) may increase or decrease consumers’ surplus.

**Proof.** See Appendix L. ■

Depending on the SPE selected, consumers and firms may be better off or worse off compared to the case with no lotteries. In an SPE with a low (high) lottery price, consumers can be better (worse) off and firms worse (better) off. We show in Appendix L that for \( k = 2 \) and \( h = 0.1 \), consumers are better off if and only if \( p_L < 0.27 \). Firms are better off if and only if \( p_L > 0.28 \). Notice that if \( 0.27 < p_L < 0.28 \), both firms and consumers end up worse off than in the absence of lotteries.

We have shown, that there are SPE with lotteries even though firms would be better off if lotteries were not sold. The reason is that when there are three firms, none of them can block the appearance of a lottery. The outside option is not a market without lotteries, since a lottery over the other two substitute goods can be created. In this case, the introduction of lotteries might lead to a transfer of surplus from firms to consumers. However, the assumption that the intermediaries are competitive is important for this result. If there were an intermediary with market power, it could itself capture some of the surplus from the firms, and consumers would be less likely to benefit.

## 6 Conclusion

In this paper we show that a multiproduct monopolist uses lotteries over its goods to maximize its profits. The optimal selling strategy depends on the shape of transportation costs. If the transportation costs function is concave, the multiproduct monopolist offers only one lottery with probability \( \frac{1}{2} \). However, if the transportation costs function is convex, the optimal selling strategy includes a region of consumers, relatively indifferent between the goods, in which the monopolist offers a continuum of type contingent lotteries.

We also examine the use of lotteries in oligopolistic markets. We consider first a market with only two firms. We show that if the market is fully covered, lotteries are not used in equilibrium.
under certain common conditions. However, in an uncovered market, lotteries can be sold in equilibrium and the introduction of lotteries may lead to an increase in the social welfare.

With more than two firms, no firm has the power to veto the creation of lottery, since a lottery can always be created with other firms’ products. We show that, in this case, intermediaries might be able to sell lotteries even in a fully covered market. Interestingly, firms may be worse off than in the case where no lotteries are provided.

7 Appendices

7.1 Appendix A

Proof of Proposition 1

Proof. When \( c(\cdot) \) is convex each lottery is priced in order to extract full surplus from the type with the highest utility from its consumption. Hence, the IR constraint of the type with the highest utility from each lottery pins down the lottery’s price. Given any lottery \((q, r)\) such that \(q + r < 1\), increasing the probability of getting the least preferred good \(r\) to \(r'\) has two effects. First, all types get higher utility from the lottery \((q, r')\) compared to lottery \((q, r)\).

\[
\frac{\partial}{\partial r} [q(V - c(x)) + r(V - c(1 - x))] = V - c(1 - x) > 0 \forall x
\]

Second, the type with the highest utility from lottery \((q, r')\) is closer to the midpoint \(x = \frac{1}{2}\) compared to the type with the highest utility from lottery \((q, r)\).

\[
\frac{\partial}{\partial x} q(V - c(x)) + r(V - c(1 - x)) = 0
\]

for \(x = g(r)\) where

\[
\frac{\partial}{\partial x} q(V - c(g(r))) + r(V - c(1 - g(r))) = 0
\]

\[
g'(r) = -\frac{\partial^2}{\partial x^2} q(V - c(x)) + r(V - c(1 - x))
\]

\[
-\frac{\partial}{\partial x} q'c'(x) + rc'(1 - x)
\]

\[
= -\frac{\partial}{\partial x} q'c'(x) + rc'(1 - x)
\]

\[
= -\frac{c'(1 - x)}{q'c''(x) + rc''(1 - x)}
\]

\[
g'(r) > 0
\]

given that \(c(\cdot)\) is convex. The implication is that the lotteries can be priced higher. This is true for any \(r'\), until \(r' = 1 - q\).

When \(c(\cdot)\) is concave, the downward incentive compatibility constraints are the ones binding in determining the lotteries’ prices, whereas the individual rationality constraint binds only for the lowest type. The increase of \(r\) to \(r'\) has again two effects. First, as in the convex case, all types get a higher utility from \((q, r')\). Second, in relative terms, the lowest types are the ones that gets the biggest utility gain from an increase in the probability of getting the least preferred
good.
\[
\frac{\partial^2}{\partial x \partial r} q (V - c (x)) + r (V - c (1 - x))
\]
\[c' (1 - x) > 0\]
This implies that the downward incentive compatibility constraint are relaxed by an increase of \( r \) and the lotteries can be priced higher. ■

7.2 Appendix B

We show in this appendix that if \( x \in \left[\frac{1}{2}, 1\right] \), then the optimal solution implies \( q(x) \geq \frac{1}{2} \). Suppose an hypothetical optimal mechanism where \( q(x') < \frac{1}{2} \) for some \( x' > \frac{1}{2} \). There are two possible situations. First, \( q(x) \) is weakly increasing everywhere. In this case we must have that \( q(x) < \frac{1}{2} \) for all \( x < x' \). However, if apply the Mirrlees technique in the segment \([0, x']\) we show as in section 3.2 that \( q \left( \frac{1}{2} \right) = 1 \), which is a contradiction.

The second situation is when \( q(x) \) is strictly decreasing somewhere. This cannot happen because it violates the IC of at least one type. If \( q(x) \) is strictly decreasing somewhere, we must have two types \( y \) and \( z \), in the same neighborhood, such that \( y > z \) and \( q(y) < q(z) \). Suppose without loss of generality that \( y > z > \frac{1}{2} \). This implies that

\[
U(y, q(z)) > U(y, q(y))
\]

Hence, to satisfy type \( y \)'s I.C. constraint we must have

\[
p(z) - p(y) > U(y, q(z)) - U(y, q(y))
\]

(23)

Notice that \( y > z \) and \( q(y) < q(z) \) imply

\[
U(y, q(z)) - U(y, q(y)) > U(z, q(z)) - U(z, q(y))
\]

(24)

Conditions (23) and (24) imply that we must have

\[
p(z) - p(y) > U(z, q(z)) - U(z, q(y))
\]

(25)

However, condition (25) violates the I.C. constraint of type \( z \). Hence, if \( y > z \) and \( q(y) < q(z) \) it is not possible to satisfy both type \( y \) and type \( z \) I.C. constraints.

7.3 Appendix C

Proof of Proposition 6 (continued)

When \( c(x) \) is convex, the utility of a lottery is not a monotonic function of \( x \). Still, for a given lottery \( q \), consumers can be ranked in terms of the utility they can get from that lottery: We call the lowest type for a lottery \( q \) the type of consumer \( x \) that gets the lowest utility from that lottery. Any lottery used in the optimal mechanism is priced in such a way that the IR constraint of its lowest type binds.\(^8\) Let \( x^{**} \) denote this type. Assume that in equilibrium type \( x^{**} \) buys lottery \( q(x^{**}) \) at price \( p(x^{**}) \) so that \((1 - q(x^{**})) (V - c(x^{**})) + q(x^{**}) (V - c(1 - x^{**})) = p(x^{**}) \). Next

\(^8\) Notice that, if \( f \) is concave, the lowest type for any lottery \( q \) is \( x = \frac{1}{2} \). If \( f \) is convex, the lowest type is a function of \( q \); it varies with the lotteries we consider.
we prove by contradiction that, in that case, the IR constraints of all types $\frac{1}{2} \leq x \leq x^{**}$ have to bind.

Let’s assume that there is a type $\tilde{x}$, such that $\frac{1}{2} \leq \tilde{x} < x^{**}$, for which the IR constraint does not bind. Consider the case in which $\tilde{x}$ buys a lottery $q'$ such that $q' < q(x^{**})$. Given that we are assuming that $\tilde{x}$’s IR constraint does not bind, the price of that lottery has to be $p(q') = V - (1 - q')c(\tilde{x}) - q'c(1 - \tilde{x})$. Furthermore, let’s assume that type $\tilde{x}$ is the type that gains the highest utility from $q'$, in other words $q' = q(\tilde{x})$. Given that $q' < q(x^{**})$, there is a type $x''$, $\frac{1}{2} \leq x'' < \tilde{x}$, such that $p(q') = V - (1 - q')c(x'') - q'c(1 - x'')$. Moreover, it has to be that $p(q') \geq V - (1 - q')c(x^{**}) - q'c(1 - x^{**})$. Otherwise, type $x^{**}$’s constraints would be violated. If $p(q') > V - (1 - q')c(x^{**}) - q'c(1 - x^{**})$, then there would be some types in the neighborhood of $x^{**}$ that do not buy lottery $q'$. For the sake of simplicity, and without loss of generality, let us consider the case in which $p(q') = V - (1 - q')c(x^{**}) - q'c(1 - x^{**})$.

Notice that $p(q')$ is a increasing function of $q'$ and an increasing function of $x^{**}$ for any $x^{**} > \tilde{x}$. Moreover, $x''$ is an increasing function of $p$ (ergo a increasing function of $q'$).

Given that $V - (1 - q')c(x^{**}) - q'c(1 - x^{**}) = V - (1 - q')c(x'') - q'c(1 - x'')$, type $x''$ can be expressed as the solution of the equation $q'(c(1 - x'') - c(1 - x^{**})) = (1 - q')c(x^{**}) - c(x'')$, or, alternatively, $x'' = g(q')$.

Now, for any given interval of types $[x'', x^{**}]$, let us compare the expected revenue of a mechanism $\mu^A$ with $q(x)$ and $p(x)$ such that IC and IR bind for all types $(q = \frac{c'(1-x)}{c'(1-x) + c'(x)}$ and $p(q) = V - (1 - q) + q^2$), and the expected revenue of a mechanism $\mu^B$ in which the types buy the same lottery $q'$ at price $p(q')$.

The expected revenue from the first mechanism is

$$ER(\mu^A) = \int_{x''=g(q')}^{x^{**}} \left\{ V - \left( \frac{c'(1-x)}{c'(1-x) + c'(x)} \right) + \left( \frac{c'(1-x)}{c'(1-x) + c'(x)} \right)^2 \right\} dx$$

The expected revenue from the second mechanism is

$$ER(\mu^B) = (x^{**} - x'') \left( V - (1 - q')c(x'') - q'c(1 - x'') \right)$$

or equivalently

$$ER(\mu^B) = (x^{**} - g(q')) \left( V - (1 - q')c(x^{**}) - q'c(1 - x^{**}) \right)$$

Define $x''$ as $p(x'') = p(q')$. In other words $x''$ is the type such that the lottery $q(x'')$ is priced the same as the lottery $q'$. Notice that $x'' < x^{**}$. Otherwise type $x^{**}$’s IC constraint would be violated. Notice also that $x'' > x''$, otherwise $x''$ would be the type with the highest utility from consuming lottery $q'$ and there would not exist a type $\tilde{x}$ that buys lottery $q'$ and have some positive utility (i.e. the IR does not bind). In the interval $[x'', x^{**}]$, mechanism $\mu^A$ gives higher expected revenue than mechanism $\mu^B$; the opposite is true in $[x'', x''']$.

In summary, the gain is equal to

$$\int_{x''}^{x^{**}} \left\{ V - \left( \frac{c'(1-x)}{c'(1-x) + c'(x)} \right) + \left( \frac{c'(1-x)}{c'(1-x) + c'(x)} \right)^2 \right\} dx - (x^{**} - x'') \left( V - q'c(x'') - (1 - q')c(1 - x'') \right)$$

This is the only relevant case.
and the loss is
\[
(x^m - x^m) (V - (1 - q)c(x^m) - qc(1 - x^m)) \]
\[
= \int_{x^m}^{x^m} \left\{ V - \left( \frac{c'(1 - x)}{c'(1 - x) + c'(x)} \right) \right. \]
\[
\left. + \left( \frac{c'(1 - x)}{c'(1 - x) + c'(x)} \right)^2 \right\} \, dx
\]

If the function \( \frac{d^2}{dx^2} \left( V - \frac{c'(1 - x)}{c'(1 - x) + c'(x)} c(x) - \frac{c'(x)}{c'(1 - x) + c'(x)} c(1 - x) \right) \geq 0 \) in a neighborhood of \( x = \frac{1}{2} \), then it’s immediate to verify that the loss is bigger than the gain for any \( q \) if \( x^m \) is close enough to the neighborhood of \( x = \frac{1}{2} \) where the function \( V - \frac{c'(1 - x)}{c'(1 - x) + c'(x)} c(x) - \frac{c'(x)}{c'(1 - x) + c'(x)} c(1 - x) \) is convex.

For consumers located in the segment \([x^m, 1]\) only their IC constraint bind. Hence, in this region the price of a lottery with probability \( q(x) \) of getting good A is
\[
p(x) = V - [(1 - q)c(x) + qc(1 - x)] - \int_{x^m}^{x} \left\{ -(1 - q)c'(x) - qc'(1 - x) \right\} \, dz
\]
The monopolist maximizes
\[
\pi = \int_{\frac{1}{2}}^{1} p(x) \, dx
\]
Which is given by
\[
\max_{x^m, q} \int_{\frac{1}{2}}^{x^m} \left\{ V - \frac{c'(1 - x)c(x)}{c'(x) + c'(1 - x)} - (1 - \frac{c'(1 - x)}{c'(x) + c'(1 - x)}) c(1 - x) \right\} \, dx
\]
\[
+ \int_{x^m}^{1} \{ V - [(1 - q)c(x) + qc(1 - x)] \} \, dx
\]
\[
- \int_{x^m}^{1} \int_{x^m}^{x} \{ c'(1 - z) - q(c'(z) + c'(1 - z)) \} \, dz
\]
After simplifying and integrating by parts, we obtain
\[
\max_{x^m, q} \int_{\frac{1}{2}}^{x^m} \left( V - \frac{c'(1 - x)c(x) + c'(x)c(1 - x)}{c'(x) + c'(1 - x)} \right) \, dx
\]
\[
+ \int_{x^m}^{1} \{ V - [(1 - q)c(x) + qc(1 - x)] - (c'(1 - x) - q(c'(x) + c'(1 - x))) (1 - x) \} \, dx
\]
Notice that now only the second integral depends on \( q(x) \). Maximizing the term under the second integral with respect to \( q(x) \) for all \( x > x^m \), we obtain
\[
-c(x) + e(1 - x) + [c'(x) + c'(1 - x)] (1 - x)
\]
Expression (26) does not depend on \( q \). Hence, for each \( x \in [x^m, 1] \) the expression is either positive or negative. If, for a given \( x \), it is positive, then \( q(x) = 1 \). If it is negative, then \( q(x) \) is

\(^{10}\)For example, if \( f(x) = x^2 \), then 9 is convex over all support \([\frac{1}{2}, 1]\).
equal to the highest possible \( q \), \( q = q(x^{**}) \). The first-order condition with respect to \( x^{**} \) is

\[
V - \frac{c'(1-x^{**})c(x^{**}) + c'(x^{**})c(1-x^{**})}{c'(x^{**}) + c'(1-x^{**})} - V + (1-q(x^{**}))c(x^{**}) = 0
\]

(27)

We guess and verify that expression (26) is negative for all \( x \in [x^{**}, 1] \), and is equal to zero for \( x = x^{*} \). In that case, \( q(x) = 0 \) for all \( x \in [x^{**}, 1] \), and equation (27) simplifies to

\[
1 - x^{**} = \frac{1}{c'(1-x^{**})} \frac{c'(1-x^{**})c(x^{**}) + c'(x^{**})c(1-x^{**})}{c'(x^{**}) + c'(1-x^{**})} - \frac{c(1-x^{**})}{c'(1-x^{**})}
\]

\[
\Leftrightarrow 1 - x^{**} = \frac{c(x^{**}) - c(1-x^{**})}{c'(x^{**}) + c'(1-x^{**})}
\]

(28)

If we evaluate expression (26) at \( x = x^{**} \) and substitute \( 1 - x^{**} \) by expression (28) we verify that expression (26) is indeed equal to zero.

### 7.4 Appendix D

**Proof of Proposition 8 and Proposition 11**

Consider firm \( A \)'s profit maximization problem when the market is not fully covered and thus the marginal type \( x \) is indifferent between buying good \( A \) and not buying any good. The indifference condition of type \( x \) is

\[
V - c(x) - p_A = 0
\]

\[
x = c^{-1}(V - p_A)
\]

Firm B’s problem is symmetric. Hence, firm \( i \)'s demand is

\[
D_i = c^{-1}(V - p_i), \text{ for } i = \{A, B\}
\]

Firm \( i \)'s maximization problem is given by

\[
\max_{p_i} \pi_i = p_i * (c^{-1}(V - p_i))
\]

(29)

Hence, firm \( i \)'s optimal price is

\[
p_i(V) = \frac{c^{-1}(V - p_i)}{c^{-1}((V - p_i))}, \text{ for } i = \{A, B\}
\]

The lowest value \( V^* \) for which the market is fully covered is determined by the indifference condition of the marginal type \( x \), when \( x = \frac{1}{2} \). In that case, the consumer located at \( x = \frac{1}{2} \) is indifferent between buying any of the two goods or no good.

\[
V^* - c\left(\frac{1}{2}\right) - p_i(V^*) = 0
\]

(30)

**Proof of part b) of proposition 12**

A lottery over the two goods is sold in equilibrium at price \( p_L = p^* \) if both firms have the following strategy:
- At time $t = 1$
  The firm accepts to sell through a lottery with price $p_L$ if and only if $p_L = p^*$. The price $p^*$ satisfies the following two conditions

$$p^* \leq V - c\left(\frac{1}{2}\right)$$

(31)

and

$$p_L \left[1 - g^{-1}(2(p_L - p_i)) - g^{-1}(2(p_L - p_j))\right] > 2p_i \left[c^{-1}(V - p_i) - g^{-1}(2(p_L - p_i))\right]$$

- At time $t = 2$, the firm chooses its own price equal to

$$p_i = \frac{g^{-1}(2(p_L - p_i))}{2g^{-1}(2(p_L - p_i))} + \frac{p_L}{2}, \quad i = \{A, B\}$$

(32)

Condition (31) implies that all consumers buy either the lottery or the good directly from the firms. Each firm’s optimal price is given by condition (32) and each firm $i$’s profit is given by expression (13))

$$\pi_i = p_i g^{-1}(2(p_L - p_i)) + \frac{p_L}{2} \left[1 - g^{-1}(2(p_L - p_i)) - g^{-1}(2(p_L - p_i))\right]$$

(33)

$$i = \{A, B\}$$

If the firm deviates and does not sell through the lottery at price $p_L = p^*$, then no lottery is sold in the market at time $t = 2$. Given that we assume that $V < V^*$, the resulting market is not fully covered and by proposition 11 firm $i$’s profit is given by

$$\pi^d_i = p_i \left(c^{-1}(V - p_i)\right)$$

(34)

Where $p_i$ is given by expression (17).

Firm $i$ does not want deviate if the profit from deviating, given by expression (34), is lower than the profit given by condition (33).

**Example with** $c(x) = x$. To show that a SPE with lotteries may exist, we consider a particular case of linear transportation costs where $c(x) = x$ and $0.5 < V < 1.5$. The equilibrium price of the good in the absence of lotteries is $p_i = 1$ and $V^* = 1.5$.

**Equilibrium profit:** In this case $g(x) = c(x) - c(1 - x) = 2x - 1$. Hence,

$$g^{-1}(y) = \frac{y + 1}{2}$$

(35)

Consider the SPE with a lottery price $p_L$. Condition (31) implies that

$$p_L \leq V - \frac{1}{2}$$

(36)

Applying (35) to the expression (32) of firm $i$’s optimal price leads to

$$p_i = \frac{3}{4}p_L + \frac{1}{4}, \quad i = \{A, B\}$$

(37)
Applying (35) to the expression firm \(i\)'s demand, given by (12), we obtain

\[
D_i = \frac{2(p_L - p_i) + 1}{2}, \quad i = \{A, B\}
\]

(38)

Using expressions (37) and (38) on the expression (33) of firm \(i\)'s profit we obtain

\[
\pi_L (p_L) = 0.5p_L - 0.0625p_A^2 + 0.0625
\]

(39)

**Deviation profit:** If firm \(i\) deviates and does not sell the good through the lottery, no lotteries are offered in the market. Each firm’s optimal price is \(p^{NL} = \frac{V}{2}\), and each firm’s profit is given by \(\pi = \frac{V}{4}\).

Firm \(i\) does not deviate if \(\pi_L (p_L) > \pi\). This implies

\[
p_L \geq 4 - 2\sqrt{4.25 - V^2} \quad \text{and} \quad V > 0.5
\]

(40)

Concluding, conditions (36) and (40) imply that there is a SPE with lotteries if the lottery price \(p_L\) is such that

\[
4 - 2\sqrt{4.25 - V^2} \leq p_L \leq V - 0.5 \quad \text{and} \quad 0.5 < V < 1.5
\]

(41)

### 7.5 Appendix E

**Proof of Proposition 10 (continued)**

Let \(d(p_L, p_i)\) denote the first order condition of firm \(i\)'s profit maximization problem, given by equation (15)

\[
d(p_L, p_i) = p_i - \frac{1}{2} \left( \frac{g^{-1}(2(p_L - p_A))}{g^{-1r}(2(p_L - p_A))} + p_L \right) = 0
\]

We define a function \(p_A = h(p_L)\) such that \(d(p_L, h(p_L)) = 0\). From the Implicit Function Theorem we obtain

\[
h'(p_L) = -\frac{\partial d}{\partial p_L} = -\frac{1}{2} \left( \frac{2[g^{-1r}(\cdot)][g^{-1}(\cdot)] - 2[g^{-1r}(\cdot)][g^{-1}(\cdot)] + 1}{1 - \frac{1}{2} \left( -2[g^{-1r}(\cdot)][g^{-1}(\cdot)] + 2[g^{-1r}(\cdot)][g^{-1}(\cdot)] \right)} \right)
\]

\[
= \frac{3}{2} - \frac{[g^{-1r}(\cdot)][g^{-1}(\cdot)]}{[g^{-1r}(\cdot)][g^{-1}(\cdot)]}
\]

which implies that \(h'(p_L) \geq 0\), if and only if \(\frac{[g^{-1r}(\cdot)][g^{-1}(\cdot)]}{[g^{-1r}(\cdot)][g^{-1}(\cdot)]} \geq 2\) or \(\frac{[g^{-1r}(\cdot)][g^{-1}(\cdot)]}{[g^{-1r}(\cdot)][g^{-1}(\cdot)]} \leq \frac{3}{2}\). Notice that \(h'(p_L) < 1\)

### 7.6 Appendix F

**Proof of Proposition 13**

Consider the same example as in Appendix D: \(c(x) = x\) and \(0.5 < V < 1\).

a) **Welfare in the absence of lotteries.**
Following proposition 11, firm’s optimal price and output are respectively \( p_{NL} = \frac{V}{2} \) and \( x = \frac{V}{2} \). Firms’ total profit is \( 2\pi = \frac{V^2}{2} \) and the consumer surplus is \( CS = \frac{V^2}{4} \). Hence, the social surplus is \( SS = \frac{3V^2}{4} \).

b) Welfare with lotteries.

b1) Firms’s profits: as derived in Appendix D, firms’ profit, \( \pi_L (p_L) \), is given by expression (39).

b2) Consumer Surplus: the consumer surplus is the sum of the surplus of consumers who buy the good and the surplus of consumers who buy the lottery

\[
CS = \int_0^{x^*} (V - p_i - x) \, dx + \int_{x^*}^{1-x^*} \left( V - p_L - \left( \frac{x}{2} + \frac{1-x}{2} \right) \right) \, dx + \int_{1-x^*}^{1} (V - p_i - (1-x)) \, dx
\]

Replacing \( p_i \) and \( x^* \) by expressions (37) and (38) respectively, we obtain

\[
CS = V - 0.875 p_L + 0.0625 p_L^2 - 0.4375
\]

(42)

b3) Social Surplus: the social surplus is the sum of firms’ profits and consumer surplus

\[
SS = \pi(p_L) + CS = V + 0.125 p_L - 0.0625 p_L^2 - 0.3125
\]

(43)

Part i) of proposition 13: the social welfare with lotteries is higher than the social welfare in the absence of lotteries if

\[
1 - 2\sqrt{4V - 3V^2 - 1} < P_L \quad \text{and} \quad V \geq \frac{5}{6}
\]

or if

\[
V < \frac{5}{6} \quad \text{and} \quad 0 < P_L \leq V - 0.5
\]

(44)

(45)

Combining condition (41) for the existence of an equilibrium with lotteries with conditions (44), (45), we obtain the conditions that guarantee existence of a SPE with lotteries with a higher social welfare than the SPE without lotteries

\[
\max \left\{ 1 - 2\sqrt{4V - 3V^2 - 1}, 4 - 2\sqrt{4.25 - V^2} \right\} < P_L \leq V - 0.5 \quad \text{and} \quad \frac{5}{6} \leq V \leq 1
\]

or

\[
4 - 2\sqrt{4.25 - V^2} < P_L \leq V - 0.5 \quad \text{and} \quad 0.5 < V < \frac{5}{6}
\]

Part ii) of proposition 13: In any SPE with lotteries both firms’ profits must be weakly higher than in the SPE without lotteries, otherwise each firm would deviate and would not sell its good through the lottery.

Part iii) of proposition 13: The consumer surplus in presence of lotteries, given by condition (42), is higher than in a market without lotteries, given by \( CS = \frac{V^2}{4} \), if

\[
P_L < 7 - 2\sqrt{V^2 - 4V + 14}
\]

(46)

Condition (46) together with the necessary condition to have an SPE equilibrium with lotteries, condition (41), are necessary and sufficient for the existence of a SPE with lotteries with a higher consumer surplus than in the SPE without lotteries.
\[ 4 - 2\sqrt{4.25 - V^2} \leq P_L < 7 - 2\sqrt{V^2 - 4V + 14} \]

\[ P_L \leq V - 0.5 \]

\[ 0.5 \leq V \leq 1 \]

### 7.7 Appendix G

**Lemma 17** Demand function in the MS model in the absence of lotteries. If the price of good \( j \) and \( z \) are \( p_j = p_z = w \), and \( 0 < h \leq \frac{1}{3} \), the demand of good \( i \) is given by the following expression, for any \( i \in \{A, B, C\} \).

\[
D_i = \begin{cases} 
0 & p_i > w + 1/3 \\
\frac{1}{3} + w - p_i + \frac{2}{6} - \frac{2}{k+1} \left( \frac{1}{6} + h + \frac{k}{6} + p_i - w \right) & w - h < p_i < w + 1/3 \\
\frac{2}{3} + 2 \left( \frac{1}{6} + \frac{1}{6} + h + \frac{1}{6}k + p_i - w \right) & w - h - (k-1)\frac{1}{6} < p_i < w - \frac{1}{3} \\
\frac{1}{3} + w - p_i + \frac{2}{6} - \frac{2}{k+1} \left( \frac{1}{6} + h + \frac{k}{6} + p_i - w \right) & p_i < w - h - (k-1)\frac{1}{6} 
\end{cases}
\] (47)

We only consider firm \( i \)'s demand when firms \( j \) and \( z \) charge the same price, \( p_j = p_z = w \), because it is the relevant part of the demand to verify the existence of a symmetric Nash equilibrium.

The first two sub-segments of the demand, when \( p_i > w - h \), are exactly as in the Salop model.

If \( p_i < w - h \), firm \( i \) also attracts consumers in the segment \( jz \). Consider the sub-segment \( jz \) and let \( y \) be the distance of a consumer in this segment to good \( j \). A consumer is indifferent between good \( i \) and \( j \) iff

\[
V - \left[ \frac{1}{6} + k \left( \frac{1}{6} - y \right) + h \right] \tau - p_i = V - y - w
\]

which yields

\[
y = \frac{1}{1+k} \left( p_i - w + \left( h + \frac{1}{6}k + \frac{1}{6} \right) \right), \quad 0 \leq y \leq \frac{1}{6}
\] (48)

The demand for good \( i \) from the segment \( jz \), \( D_i^{jz} \), is given by

\[
D_i^{jz} = 2 \left( \frac{1}{6} - y \right) \Rightarrow D_i^{jz} = 2 \left( \frac{1}{6} - \frac{1}{k+1} \left( \frac{1}{6} + k + p_i - w \right) \right)
\]

where we replaced \( y \) by expression (48).

Then, the total demand for good \( i \) if \( w - \frac{1}{3} < p_i < w - h \) is

\[
D_i = D_i^{ij} + D_i^{iz} + D_i^{jz} = \frac{1}{3} + w - p_i + 2 \left( \frac{1}{6} - \frac{1}{k+1} \left( \frac{1}{6} + h + \frac{1}{6}k + p_i - w \right) \right) \quad \text{if } w - \frac{1}{3} < p_i < w - h
\]

If \( w - h - (k-1)\frac{1}{6} < p_i < w - \frac{1}{3} \), then \( D_i^{ij} = D_i^{jz} = \frac{1}{3} \). If \( p_i < w - h - (k-1)\frac{1}{6} \), then the demand for good \( i \) is equal to the whole market.

27
Proof of Proposition 14

The symmetric equilibrium in the standard Salop model is such that firm $i$‘s optimal price is $p_i = \frac{1}{3}$, $i = \{A, B, C\}$ and each firm’s profit is equal to $1/9$.

The only difference between the standard Salop model and MS model is the specification of the utility for the least preferred good. Hence, the two models are equivalent when we consider small price deviations from a symmetric equilibrium. Indeed, for small price deviations such that $w - h < p_i < w + 1/3$, only the utility for the two most preferred goods is relevant.

Let us assume that $w = \frac{1}{3}$ as in the standard Salop symmetric equilibrium. To check if this is a symmetric equilibrium also in the MS model, we need to verify that a large deviation, such that $p_i < \frac{1}{3} - h$, is not optimal. Using the demand function in the absence of lotteries, derived in Lemma 17, we obtain that firm $i$’s profit function is given by

$$\pi_{d_i} = \left(1 + \frac{1}{3} - p_i\right) p_i + p_i \left(\frac{2}{6} - \frac{2}{k+1} \left(\frac{1}{6} + h + \frac{1}{6} k + p_i - \frac{1}{3}\right)\right)$$

if $\frac{1}{3} - h - (k - 1) \frac{1}{6} < p_i < \frac{1}{3} - h$

Firm $i$’s optimal price is

$$p_i = \frac{\frac{4}{3} - 2h + \frac{2k}{3}}{6 + 2k}$$

Firm $i$’s profit if it deviates is given by

$$\pi = \left(1 + \frac{1}{3} - \frac{\frac{4}{3} - 2h + \frac{2k}{3}}{6 + 2k}\right) \left(\frac{\frac{4}{3} - 2h + \frac{2k}{3}}{6 + 2k}\right) +$$

$$+ 2 \left(\frac{\frac{4}{3} - 2h + \frac{2k}{3}}{6 + 2k}\right) \left(\frac{1}{6} - \frac{1}{k+1} \left(\frac{1}{6} + h + \frac{1}{6} k + \frac{\frac{4}{3} - 2h + \frac{2k}{3}}{6 + 2k} - \frac{1}{3}\right)\right)$$

(49)

The firm does not deviate if the profit in expression (49) is less than the equilibrium profit, $1/9$. This occurs if and only if

$$k > \frac{9h^2 + 1 - 12h}{6h} \text{ or } h \geq 0.0893$$

(50)

If condition (50) is verified, the symmetric equilibrium in the MS and Salop models is the same.

7.8 Appendix H

Determination of firm $i$’s demand in the presence of a lottery among three firms, $L_3$, sold at a price $p_{L}$. For simplicity, we only consider only part of the demand function by assuming that:

i) $p_j = p_z = w$

ii) $w > p_L + h$.

iii) $w < p_L + \frac{1}{3} + \frac{h}{2} + \frac{h}{3}$

Assumption i) is the same made in the appendix G. Assumption ii) implies that consumers close enough to the middle of the segment $jz$ prefer the lottery $L_3$ than each of these two goods. Assumption iii) implies that consumers at the extreme of the sub-segment $jz$ prefer one of these goods over the lottery.
Lemma 18 Under assumptions i), ii), and iii), the demand for good i, where i = \{A, B, C\}, in a market with a three goods lottery sold at price p_L is the following:

\[
D_i(p_i, p_L) = \begin{cases} 
0 & p_i \geq p_L + \frac{1}{6} + \frac{1}{6}h + \frac{h}{k} \\
2\left(\frac{1}{6} + \frac{1}{3}k \right)(p_L - p_i) + \frac{1}{k}\left(\frac{h}{k} + k\right) & p_L - h \leq p_i < p_L + \frac{1}{6} + \frac{1}{6}h + \frac{h}{k} \\
2\left(\frac{1}{6} + \frac{1}{3}k \right)(p_L - p_i) + \frac{h}{k} & p_L - \frac{1}{6} + \frac{1}{6}h + \frac{h}{k} \leq p_i < p_L - h \\
\frac{2}{3} + 2\left(\frac{1}{6} - \frac{1}{3}k \right)\left(\frac{2}{3}h + \frac{1}{6}k - p_L + p_i\right) & -\frac{2}{3}k + p_i \leq p_L - \frac{1}{6} + \frac{1}{6}h + \frac{h}{k} \\
\frac{2}{3} + 2\left(\frac{1}{6} - \frac{1}{3}k \right)\left(\frac{2}{3}h + \frac{1}{6}k - p_L + p_i\right) & -\frac{2}{3}k + p_i \leq p_L - \frac{1}{6} + \frac{1}{6}h + \frac{h}{k} \\
\frac{2}{3} + 2\left(\frac{1}{6} - \frac{1}{3}k \right)\left(\frac{2}{3}h + \frac{1}{6}k - p_L + p_i\right) & -\frac{2}{3}k + p_i \leq p_L - \frac{1}{6} + \frac{1}{6}h + \frac{h}{k} \\
\end{cases}
\]

(51)

In the second section of the demand, when \(p_L - h \leq p_i < p_L + \frac{1}{6} + \frac{1}{6}h + \frac{h}{k}\), firm i’s price is such that it attracts consumers only from segments ij and iz. To obtain this section of the demand function, consider a consumer located in the segment ij and let \(x_i\) denote the distance of this consumer to good i. The consumer located at \(x_i\), indifferent between buying good i and the lottery is given by

\[V - x_i - p_i = V - \frac{1}{3}x_i - \frac{1}{3}\left(\frac{1}{6} + \frac{1}{6} - x_i\right) - \frac{1}{3}\left(\frac{1}{6} + h + k\left(\frac{1}{6} - x_i\right)\right) - p_L\]

(52)

Solving for \(x_i\), we obtain the second section of the demand function

\[D_i = 2x_i = 2\left(\frac{1}{6} + \frac{3}{k+3} (p_L - p_i) + \frac{h}{k+3}\right)\]

(53)

If \(p_i \geq p_L + \frac{1}{6} + \frac{1}{6}h + \frac{h}{k}\), expression (53) becomes negative and hence firm i’s demand is zero.

When \(p_i < p_L - h\), firm i also attracts consumers in the segment jz. Thus, we add the demand that comes from this segment to expression (53) to obtain firm i’s demand. In the segment jz, the consumer located at \(y_i\) is indifferent between buying from firm i and buying \(L_3\) if

\[V - \left(\frac{1}{6} + h + k\left(\frac{1}{6} - y_i\right)\right) - p_i = V - \frac{1}{3}y_i - \frac{1}{3}\left(\frac{1}{3} - y_i\right) - \frac{1}{3}\left(\frac{1}{6} + h + k\left(\frac{1}{6} - y_i\right)\right) - p_L\]

which yields

\[y_i = \frac{3}{2k} \left(\frac{2}{3}h + \frac{1}{9}k - p_L + p_i\right) \quad 0 \leq y_i \leq \frac{1}{6}\]

(54)

Firm i’s demand coming from this segment is

\[D_i^{jz} = 2\left(\frac{1}{6} - y_i\right) \Rightarrow D_i^{jz} = \left(\frac{3}{2} - \frac{3}{k}\right)\left(\frac{2}{3}h + \frac{1}{9}k - p_L + p_i\right)\]

(55)

Adding expression (53) and (55) yields the firm i’s total demand when \(p_i < p_L - h\).

The last two sections of the demand function, given by (51) occur when \(p_i\) is low enough such that all consumers in one segment buy from firm i.

Proof of Proposition 15

Part a) There is a SPE without lotteries if each firm has the following strategy:
- at time \(t = 1\), the firm decides not to sell its good through a lottery at any price \(p_L\)
- at time \(t = 2\), the firm chooses the price \(p = \frac{1}{3}\)
At time $t = 2$, firms choose the optimal price knowing that no lotteries are offered in the market. Hence, by proposition 14 the optimal price is $p = \frac{1}{3}$.

At time $t = 1$, no firm has an incentive to deviate and join a lottery at any price $p_L$ since lotteries are only sold if two or more firms sell their goods through the lottery.

The outcome in this equilibrium is exactly the same as in the market where lotteries are not allowed.

**Part b):** A lottery over the three goods is sold at price $p_L = p^*$ in equilibrium, if firms’ strategies satisfy the following conditions:

- at time $t = 1$
  
  i) for $p_L \neq p^*$, the firm does not sell through the lottery
  
  ii) for $p_L = p^*$, the firm accepts to sell through the lottery

where $p^*$ satisfies the following condition

$$3 + k - 6h - \frac{1}{30}\sqrt{-126h + 6k - 42hk + 36h^2 + k^2 + 9} < p^* < \frac{1}{4} + \frac{k}{12} - \frac{h}{2} \quad (56)$$

- at time $t = 2$, each firm chooses its good optimal price given by

$$p_i = \frac{1}{6}(k + 3)\left(\frac{h}{k + 3} + \frac{4p_L}{k + 3} + \frac{1}{6}\right) \quad (57)$$

Next we show that this set of strategies is a SPE.

Condition i) and ii) guarantee that a lottery with price $p_L = p^*$ is the only lottery offered in the market. At time $t=1$, firm $i$’s maximization problem is given by

$$\max_{p_i} p_iD_i(p_i, p_L) + \frac{1}{3}pLD_L(p_i, w, p_L) \quad (58)$$

Notice that firm’s optimal prices, given by (57), together with condition (56) imply that the relevant section of the demand function for good $i$ in the neighborhood of a symmetric equilibrium is given by the second interval in the demand schedule (53).

In this case the demand for the lottery is given by

$$D_L = \left(\frac{1}{3} - 2x_i\right) + \left(\frac{1}{3} - 2x_j\right) + \left(\frac{1}{3} - 2x_z\right), \text{ where } i \neq j \neq z$$

If we replace $2x_i$ by condition (53), and $2x_j$ and $2x_z$ by the equivalent expression, we obtain the demand for lottery

$$D_L(p_i, p_j, p_z, p_L) = \frac{6}{k + 3}(p_i + 2w - 3p_L - h) \quad \text{if } p_L - h \leq p_i < p_L + \frac{1}{6} + \frac{1}{3}h + \frac{k}{18} \quad (59)$$

Where $w$ is the price charged by firms $j$ and $z$.

Replacing $D_i(p_i, p_L)$ and $D_L(p_i, w, p_L)$ by (53) and (59) in expression (58), firm $i$’s maximization problem becomes

$$\max_{p_i} p_i \left[\frac{1}{3} + \frac{6}{k + 3}(p_L - p_i) + \frac{2h}{k + 3}\right] + \frac{1}{3}pL \frac{6}{k + 3}(p_i + 2w - 3p_L - h)$$

if $p_L - h \leq p_i < p_L + \frac{1}{6} + \frac{1}{3}h + \frac{k}{18}$
Firm $i$’s optimal price is given by

$$p_i = \frac{1}{6} (k + 3) \left( \frac{h}{k + 3} + \frac{4p_L}{k + 3} + \frac{1}{6} \right), \quad i = \{A, B, C\} \quad (60)$$

Expression (60) requires that the lottery over the three goods has positive demand in equilibrium, which occurs if

$$p_L \leq \frac{1}{4} + \frac{1}{12} k - \frac{1}{2} h \quad (61)$$

To guarantee that expression $p_i$ represents the optimal price we need to verify that each firm does not want to make large deviations of its price to reach other sections of the demand. Simple computations show that large deviations are also not optimal.

Replacing $p_i, D_i(p_i, p_L)$ and $D_L(p_i, w, p_L)$ in (58) we obtain firm $i$’s profit as a function of the price of the lottery ticket

$$\pi_i(p_L) = \frac{1}{3(k + 3)} \left( \frac{1}{2} + h + 4p_L + \frac{k}{6} \right) \left( \frac{k + 3}{12} + p_L + \frac{1}{12} h \right) - \frac{2p_L}{k + 3} \left( p_L + \frac{1}{2} h - \frac{k + 3}{12} \right) \quad (62)$$

At time $t = 1$, firms simultaneously choose the prices at which they are willing to sell the good through the lottery. To check if the set of strategies above is a SPE, we need to verify that firms do not want to deviate and accept to sell the good at different lottery prices.

Suppose firm $i$ deviates from the equilibrium strategy and does not choose a lottery price $p_L = p^*$. Then, there would be a lottery at price $p_L = p^*$ between goods $j$ and $z$ sold in the market. Therefore, firm $i$ would compete simultaneously against the other two firms and the lottery over the other two firms. We show next that the demand for good $i$ is as follows

$$D_i(p_i, p_L) = 2x_i = \begin{cases} \frac{1}{3} + \frac{2h + 4(p_L - p_i)}{k + 3} & p_i \geq w + \frac{h + 2(p_L - w)}{k + 3} \\ \frac{1}{3} + w - p_i & p_i < w + \frac{2h + 4(p_L - w)}{k + 3} \end{cases}$$

Where $x_i$ is the location of the consumer, in the segment between $i$ and $j$, who is indifferent between buying good $i$ or either buying good $j$ or buying the lottery (whichever provides higher utility).

The consumer $x_i$ indifferent between lottery $jz$ and good $i$ is given by:

$$V - x - p_i = V - \frac{1}{2} \left( \frac{1}{3} - x_i \right) - \frac{1}{2} \left( \frac{1}{6} + h + k \left( \frac{1}{6} - x_i \right) \right) - p_L \quad (63)$$

Solving for $x_i$, we obtain the demand for good $i$

$$D_i = 2x_i = \frac{1}{3} + \frac{2h + 4(p_L - p_i)}{k + 3} \quad (64)$$

Expression (64) is the demand of good $i$ when $p_i \geq w + \frac{h + 2(p_L - w)}{k + 3}$. In this case the indifferent consumer between the lottery and good $i$, prefers the lottery than good $j$ and hence the lottery has positive demand.

When $p_i < w + \frac{h + 2(p_L - w)}{k + 3}$, the lottery has no demand in the segments $ij$ and $iz$. In this case, $x_i$ is the indifferent consumer between good $i$ and $j$: 31
\[ V - x_i - p_i = V - \left( \frac{1}{3} - x_i \right) - p_j \]

Solving for \( x_i \), we obtain the demand of good \( i \)
\[ D_i = 2x_i = \frac{1}{3} + w - p_i \]  
(65)

where \( w \) is the price charge by the other two firms.

At time \( t = 2 \), given \( p_L \), firm \( i \) chooses its optimal price, \( p_i \). Firm \( i \)'s profit is given by

\[
\max_{p_i} \pi \Leftrightarrow \max_{p_i} p_i D_i \Leftrightarrow 
\begin{align*}
\frac{1}{3} + 2h + 4(p_L - p_i) & \quad \text{if } p_i \geq w + \frac{h + 2(p_L - w)}{k + 3} \\
p_i(\frac{1}{3} + w - p_i) & \quad \text{if } p_i < w + \frac{h + 2(p_L - w)}{k + 3}
\end{align*}
\]  
(66)

Firm \( i \)'s optimal price is given by

\[ p_i = \frac{1}{2} \left( p_L + \frac{h}{2} + \frac{k + 3}{12} \right) \]

Replacing \( p_i \) in (66) we obtain the expression of firm \( i \)'s profit when it deviates as a function of the price of the lottery

\[ \pi_d(p_L) = \left( \frac{h}{2} + p_L + \frac{k + 3}{12} \right) \cdot \left( \frac{1}{12} + \frac{1}{2} + \frac{p_L}{k + 3} \right) \]  
(67)

The profit of deviation, given by (67), is lower than the profit of not deviating, given by (62), if the following condition holds:

\[ p_L > 3 + k - 6h \]  
(68)

Notice that condition (68) is verified if the condition (56) assumed initially holds.

Hence, if conditions (56) is satisfied, there is an equilibrium where all the three firms join the lottery at time \( t = 1 \) at a price \( p_L \). We have shown that with more than two firms, lotteries can be sold even in a fully covered market.

### 7.9 Appendix I

We assume that \( k = 4 \) and \( h = 0.02 \) and show that there is a SPE with lotteries where firms set a minimal price of the lottery, \( p_i^\ast \), such that they are willing to sell at any lottery price \( p_L \geq p_i^\ast \).

In this case, the only lottery sold in equilibrium is a lottery among the three firms at a price \( p^\ast \). If firm \( i \) deviates and accepts to sell through lottery at higher prices, \( p_L \geq p_d \), with \( p^\ast < p_d \), two lotteries are offered: One lottery among two goods, \( j \) and \( k \), denoted by \( L_2 \), sold at price \( p^\ast \) and one lottery among the three goods, denoted by \( L_3 \), at price \( p_d \).

#### Determination of the demand function
- Consumers located on segment \( jk \) strictly prefer \( L_2 \) than \( L_3 \), since it has a lower price and includes their two favorite goods.
- Consumers located on the segments $ij$ and $ik$ consider buying either $L_3$ which includes one of their favorite goods, good $i$, but has a higher price. Alternatively, they may buy $L_2$, that excludes one of their favorites goods but has a lower price.

When $k > 3$, the consumer in the segment $ij$ with the higher relative preference for $L_2$ compared with the $L_3$ is the one located at $x = \frac{1}{6}$. In the remaining of appendix we consider this case, and assume that $k = 4$. Two cases arise depending on the difference between the prices of $L_2$ and $L_3$.

1) **Consider first the case where** $p_L < p_d < p_L + \frac{1}{6}h$. **Assuming** $h = 0.02$ and $p_L = 0.2$, this implies that $p_d < 0.20333$. In this case all consumers in the subsegment $ij$ or $ik$ prefer $L_3$ over $L_2$. The demand for good $i$ is given by equation (53), where we replace $p_d$ for $p_L$.

Consider now the demand function of good $j$. This demand is divided in two different components, namely consumers in the segment $ij$ and consumers in the segment $jz$.

In the segment $ij$, the consumer indifferent between $j$ and $L_3$, located at point $x_j$, is obtained in a way similar to (53)

$$x_j = \frac{1}{6} - \frac{3(p_d - p_j)}{k + 3} - \frac{h}{k + 3}$$  \hspace{1cm} (69)

In the segment $jz$, the consumer indifferent between $j$ and $L_2$, located at $y_j$, where $y_j$ measures the distance to good $j$, is given by

$$v - \frac{1}{2}y_j - \frac{1}{2}(\frac{1}{3} - y) - p_L = v - y_j - p_j$$

Solving for $y_j$

$$y_j = p_L - p_j + \frac{1}{6}$$  \hspace{1cm} (70)

In the segment $jz$, the indifferent consumer between $z$ and $L_2$, located at a distance $y_z$ of good $j$, is given by:

$$y_z = p_z - p_L + \frac{1}{6}$$  \hspace{1cm} (71)

The demand for good $j$ is given by

$$D_j = (\frac{1}{3} - x_j) + y_j \Rightarrow D_j = \frac{1}{3} + p_L - p_j + \frac{1}{k + 3}(3p_d - 3p_j + h)$$  \hspace{1cm} (72)

The total demand for $L_3$ is given by

$$D_{L3}(p_i, p_j, p_z, p_L) = \frac{1}{3} - 2x_i + (\frac{1}{3} - 2x_j) + (\frac{1}{3} - 2x_z)$$  \hspace{1cm} (73)

Where $x_z$ is the indifferent consumer between $z$ and $L_3$ in the segment $iz$. Using equations ( (53)) and (69) in expression (73) we obtain

$$D_{L3} = \frac{1}{k + 3} (6p_i + 3p_j + 3p_z - 12p_d) - \frac{4h}{k + 3}$$  \hspace{1cm} (74)

The total demand for $L_2$ is

$$D_{L2}(p_i, p_j, p_z, p_L) = y_z - y_j$$

Which yields

33
\[ D_{L2} = p_j + p_z - 2p_L \]  

(75)

At time \( t=2 \), each firm chooses its optimal price, knowing that there two lotteries sold in the market, \( L_2 \) and \( L_3 \). Firm \( i \)'s profit maximization problem is as follows

\[
\max_{p_i} p_i D_i + p_d * \frac{1}{3} * D_{L3}
\]

Substituting \( D_i \) and \( D_{L3} \) by expressions (53) and (74), we obtain

\[
\max_{p_i} p_i \left( \frac{1}{3} + \frac{6}{k+3} (p_d - p_i) + \frac{2h}{k+3} \right) + p_d * \frac{1}{3} \left( \frac{1}{k+3} (6p_i + 3p_j + 3p_z - 12p_d) - 4 \frac{h}{k+3} \right) \]  

(76)

The optimal price of firm \( i \) is

\[
p_i = \frac{1}{6} h + \frac{k + 3}{36} + \frac{2}{3} p_d \]  

(77)

The maximization problem of firm \( j \) is

\[
\max_{p_j} p_j D_j + p_d * \frac{1}{3} * D_{L3} + p_L * \frac{1}{2} * D_{L2}
\]

Replacing \( D_j \), \( D_{L3} \) and \( D_{L2} \) by expressions (72), (74), and (75) respectively, we obtain

\[
\max_{p_j} p_j \left( \frac{1}{3} + \frac{p_L - p_j}{k+3} (3p_d - 3p_j + h) \right) + p_d \left( \frac{1}{3} \left( \frac{1}{k+3} (6p_i + 3p_j + 3p_z - 12p_d) - 4 \frac{h}{k+3} \right) + p_L \frac{1}{2} (p_j + p_z - 2p_L) \right)
\]

Which yields

\[
p_j = \frac{1}{2} + \frac{6}{k^3} \left( \frac{1}{3} + \frac{3}{2} p_L + \frac{h + 4p_d}{k+3} \right) \]  

(78)

To obtain firm \( i \)'s profit of deviation, in expression (76) we substitute \( p_i \) and \( p_j \) by expression (77) and (78) respectively and take into account that \( p_j = p_z \), \( K = 4 \) and \( h = 0.02 \)

\[
\pi_i^d = 0.286p_d - 0.133p_d^2 + 0.0335 \]  

(79)

Maximizing expression (79) subject to the restriction that \( p_d \leq 0.20333 \) yields that the optimal deviation price is \( p_d = 0.20333 \) and the profit is \( \pi_i^d = 0.08615 \). If firm \( i \) does not deviate and accepts to sell for any lottery price \( p_L \geq 0.2 \), its profit, given by expression (62). Assuming that \( p_L = 0.2 \), \( K = 4 \) and \( h = 0.02 \), the profit of firm \( i \) is \( \pi_i = 0.0964 \). Hence, the firm does not want to deviate for \( p_d < p_L + \frac{1}{6} h \).

2) Consider next the case where \( p_d > p_L + \frac{1}{6} h \). Given that \( h = 0.2 \) and \( p_L = 0.2 \), we have that \( p_d > 0.20333 \).

In this case consumers close to the midpoint of \( i \) and \( j \) buy \( L_2 \) at a price \( p_L \). The demand for good \( i \) and \( j \) are given by expression (53) and (72) respectively.

Consider now the demand for the lotteries. In the segment \( ij \), the indifferent consumer between \( L_3 \) and \( L_2 \), denoted by \( x \), is given by:
\[ v - \frac{1}{3} x - \frac{1}{3} \left( \frac{1}{3} - x \right) - \frac{1}{3} \left( \frac{1}{6} + h + k \right) \left( \frac{1}{6} - x \right) \right) - p_3 = v - \frac{1}{2} \left( \frac{1}{6} + \frac{1}{6} - x \right) - \frac{1}{2} \left( \frac{1}{6} + h + k \right) \left( \frac{1}{6} - x \right) \right) - p_L \] (80)

Solving for \( x \) equation (80) becomes

\[
x_{3\_2(x<1/6)} = \frac{1}{6} + \frac{1}{(k+3)} \left( h + 6(p_L - p_3) \right), \quad \text{if } \quad x_{3\_2(x<1/6)} < 1/6
\]

and

\[
x_{3\_2(x>1/6)} = \frac{1}{6} + \frac{1}{(k-3)} \left( -h - 6(p_L - p_d) \right), \quad \text{if } \quad x_{3\_2(x>1/6)} > 1/6
\]

Consumers located at \( x \in [x_{3\_2(x<1/6)}, x_{3\_2(x>1/6)}] \) buy \( L_2 \). The demand for the lottery \( L_2 \) is given by:

\[ D_{L2} = 2(x_{3\_2(x>1/6)} - x_{3\_2(x<1/6)}) + (y_z - y_j) \]

If we substitute \( x_{L3\_L2(x<1/6)} \) and \( x_{L3\_L2(x>1/6)} \)

\[ D_{L2} = \frac{4k}{k^2 - 9} \left( -h - 6p_L + 6p_d \right) + p_z + p_j - 2p_L \] (81)

The demand for \( L_3 \) is given by

\[ D_{L3} = 2(x_{L3\_L2(x<1/6)} - x_i) \right) + (x_j - x_{L3\_L2(x>1/6)}) + (x_z - x_{L3\_L2(x>1/6)}) \]

If we substitute \( x_{L3\_L2(x<1/6)} \) and \( x_{L3\_L2(x>1/6)} \) we obtain

\[ D_{L3} = 2(\frac{1}{k+3} + \frac{1}{k-3})(h + 6(P_L - p_d)) - \frac{3}{k + 3} (4p_d - 2p_i - p_j - p_z) - \frac{4h}{k + 3} \] (82)

Firm \( i \)'s maximization profit is given by

\[ \max_{p_i} p_i D_i + p_d * \frac{1}{3} * D_{L3} \]

Replacing \( D_i \) and \( D_{L3} \) by expressions (53) and (82) we obtain

\[ \max_{p_i} p_i \left( \frac{1}{3} + \frac{6}{k+3}(p_d - p_i) + \frac{2h}{k+3} \right) + \frac{p_d}{3} \left[ \frac{2}{k+3} + \frac{2}{k-3}(h + 6(p_L - p_d)) - \frac{3}{k + 3} (4p_d - 2p_i - p_j - p_z) - \frac{4h}{k + 3} \right] \] (83)

Firm \( i \)'s optimal price is given by

\[ p_i = \frac{1}{12} + \frac{1}{6} h + \frac{2}{3} p_d + \frac{k}{36} \Rightarrow p_i = \frac{2}{3} p_d + 0.198 \] (85)

Consider now firm \( j \)'s maximization problem.
Replacing \( D_j, D_{L2}, D_{L3} \) by expressions (72), (81) and (82) respectively we obtain

\[
\max_{p_j} p_j D_j + p_d \frac{1}{3} D_{L3} + p_L \frac{1}{2} D_{L2}
\]

\[
\max_{p_j} (\frac{1}{3} - \frac{1}{6} - \frac{1}{k+3} (-h - 3(p_d - p_j)) + (p_L - p_j + \frac{1}{6}))
\]

\[
+ p_d \frac{1}{3} (\frac{2}{k+3} + \frac{1}{k-3})(h + 6(p_L - p_d)) - \frac{3}{k+3} (4p_d - 2p_i - p_j - p_z) - \frac{4h}{k+3})
\]

\[
+ p_L \frac{1}{2} (-h - 6p_L + 6p_d) + p_z + p_j - 2p_L)
\]

If we assume \( K = 4, h = 0.02 \) and \( p_L = 0.2 \), we obtain that the firm’s optimal price is given by

\[
p_j = \frac{1}{5} p_d + 0.22267
\]

Replacing \( p_i \) and \( p_j \) by expressions (85) and (86) in the expression of firm \( i \)’s deviation profit (83). Notice that in a symmetric equilibrium, \( p_j = p_z \), and that we assume that \( h = 0.02, k = 4, p_L = 0 \). In this case firm’s profit is given by

\[
\pi_i^d = p_d \left( 1.0513 - \frac{514}{105} p_d \right) + \left( \frac{2}{3} p_d + 0.19778 \right) \left( \frac{2}{7} p_d + 0.16952 \right)
\]

At time \( t=1 \), firm \( i \) chooses the optimal minimal price of the lottery \( p_d \) by maximizing expression (87) which yields \( p_d = 0.12974 \). Since in this case \( p_d \geq 0.20333 \), the optimal price in this interval is \( p_d = 0.20333 \) and the profit of deviation is \( \pi_d = 0.08725 \), which is lower than the profit when the firm does not deviate, \( \pi_i = 0.0964 \).

Notice that, in this example, the profit in the equilibrium with lotteries is lower than in the equilibrium with no lotteries, which is equal to \( \pi_i = \frac{1}{5} \).

\section{Appendix L}

\subsection*{Proof of Proposition 16}

We assume that \( k = 2, h = 0.1 \).

\textbf{Part i)} Condition (56) to have a SPE with a lotteries among the three goods is satisfied if \( 0.15 \leq p_L \leq 0.29 \).

Replacing \( k, h \) in expression (62) we obtain firm’s profit

\[
\pi(p_L) = \frac{1}{3} p_L - \frac{2}{15} p_L^2 + \frac{2}{69}
\]

The profit with lotteries is higher than the profit in the absence of lotteries, \( \pi_{NL} = 1/9 \), if and only if \( p_L > 0.28 \).

Therefore, there is an SPE with lottery price \( p_L \), \( 0.15 \leq p_L \leq 0.28 \), where firms have lower profits than in a market without lotteries; and there is an SPE with a lottery price \( p_L \), such that \( 0.28 \leq p_L \leq 0.29 \), where firms have higher profits than in a market without lotteries.

\textbf{Part ii)} Assume that \( k = 2, h = 0.1 \).
Consumers’ surplus in the SPE without lotteries.

The consumers’ surplus is equal to the consumers’ gross surplus, $V$, minus the sum of firms’ profits, $3\pi$, and consumers’ transportation costs, $TC$.

$$CS = V - 3\pi - TC$$  \hspace{1cm} (89)

Which in this case is equal to

$$CS = V - \frac{1}{3} - \frac{1}{12} = V - 0.41667$$  \hspace{1cm} (90)

Consumers’ surplus in the SPE with lotteries.

In the presence of lotteries, consumer’s transportation costs are higher than in the SPE without lotteries. This is because consumers who buy the lottery ticket do not always receive their most preferred good. In this case, the consumer’s transportation costs are given by

$$TC = 3x_i^2 + 3\left(\frac{1}{3} - 2x_i\right)\left(\frac{1}{3} + \frac{x_i}{2}\right) + \frac{1}{3} \left(\frac{1}{6} + \frac{1}{3}h - \frac{x_i}{2}\right) + \frac{1}{3} \left(\frac{1}{2} + h + \frac{k}{2}\right)$$  \hspace{1cm} (91)

Replacing the equilibrium prices, given by (60), in the condition (52), we obtain that the indifferent consumer between buying the good and the lottery is given by

$$x_i = \frac{1}{12k + 36} (12p_L - k + 6h - 3) + \frac{1}{6}$$  \hspace{1cm} (92)

Using expression (92) we can replace $x$ in equation (91)

$$TC = 0.2p_L^2 - 0.187p_L + 0.125$$  \hspace{1cm} (93)

If we replace firms’ profits, given by expression (88), and the transportation costs, given by expression (93) in the expression (89) we obtain that the consumer surplus with lotteries is

$$CS_L(p_L) = V - 0.813 p_L + 0.2p_L^2 - 0.212$$  \hspace{1cm} (94)

The consumer surplus in the SPE with lotteries, given by (94) is higher than in the SPE without lotteries, given by expression (90), if the price of the lottery is less than 0.27, $p_L < 0.27$. If instead $p_L > 0.27$, the introduction of lotteries increases the consumer surplus.

References


