Overlapping risk adjusted sets of priors and the existence of efficient allocations and equilibria with short-selling✩

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Abstract

The theory of existence of equilibrium with short-selling is reconsidered under risk and ambiguity modelled by risk averse variational preferences. No-arbitrage conditions are given in terms of risk adjusted priors. A sufficient condition for existence of efficient allocations is the overlapping of the interiors of the risk adjusted sets of priors or the inexistence of mutually compatible trades, with non-negative expectation with respect to any risk adjusted prior. These conditions are necessary when agents are not risk neutral at extreme levels of wealths. It is shown that the more uncertainty averse or risk averse the agents, the more likely are efficient allocations and equilibria to exist.

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1. Introduction

The issue of the relationship between agents’ beliefs and risk tolerances and the existence of efficient allocations and equilibria has first been considered, in the early seventies, by Grandmont [14], Green [15] and Hart [18] for markets with short-selling in the context of temporary equilibrium models and assets equilibrium models and reconsidered later by Hammond [16] and Page [25,26]. In these early models, investors were assumed to have a single homogeneous or heterogeneous probabilistic belief and be risk averse expected utility maximizers (EU). The Hammond, Hart, Page’s (HHP) model has later been generalized to a model with consumption sets unbounded below. Sufficient conditions of no arbitrage were given for existence of an equilibrium (see e.g. Allouch et al. [1], Dana et al. [7], Nielsen [24], Page et al. [27], Page and Wooders [28] and Werner [33]). The no-arbitrage conditions were shown to be necessary for existence of equilibrium under adequate assumptions. More recently, risk sharing of an aggregate capital between different investors using convex measures of risk gave rise to problems of efficiency with short-selling (see e.g. Heath and Ku [19], Dana and Le Van [8], Barrieu and El Karoui [2], Filipovic and Svindland [12] and Jouini et al. [20]).

This paper reconsiders the equilibrium theory of assets with short-selling when there is risk and ambiguity. The variational preferences axiomatized by Maccheroni, Marinacci and Rustichini [23] (denoted MMR from now on) are used. Variational preferences nest many of the models developed to study ambiguity in the decision theoretic, financial and economic literature, in particular, the maxmin expected utility of Gilboa and Schmeidler [13], the penalty preference functionals of Hansen and Sargent [17] and the convex measures of risk introduced in mathematical finance (see Föllmer and Schied [11] for an overview). A risk averse variational preference is characterized by a convex cost (penalty) function defined on the probability simplex and a concave utility index that models risk-aversion. Up to a minus sign, convex measures of risk correspond to a risk neutral agent with a zero discount rate. Without loss of generality, attention may be restricted to the probabilities with finite cost that we call the priors. To simplify as much as possible the analysis, we assume complete markets and consider a standard Arrow–Debreu model of state contingent claims.

While in the HHP model, the useful trading directions have been characterized in the early seventies as trades fulfilling the incomplete mean condition by Bertsekas [3] and Hart [18], the no-arbitrage conditions have never been made explicit. The first contribution of the paper is the characterization, for MMR preferences, of the useful trading directions, the no-arbitrage prices, the concept of collective absence of arbitrage, in terms of risk adjusted sets of priors. The second contribution of the paper is to provide under the no half-line condition, necessary and sufficient conditions for existence of efficient allocations or equilibria: a necessary and sufficient condition is either that the intersection of the interiors of the risk adjusted sets of priors is non-empty or the inexistence of mutually compatible trades with non-negative expectations with respect to any risk adjusted prior. The condition that the intersection of the interiors of the risk adjusted set of priors is non-empty generalizes the conditions given in the early seventies for single beliefs. An equilibrium does not exist if agents disagree “very much”. This happens if agents consider as relevant disjoint subsets of states of the world. For equilibrium to exist, agents must have sets of priors with overlapping supports, where by support, we mean a state of the world with a strictly positive probability for some prior. Unfortunately, even when this condition is fulfilled, there may not be an equilibrium if their sets of priors are too different. As written by Hart [18] when agents are very risk averse, strong disagreement on expectations may be compatible with the existence of an equilibrium.
The paper is organized as follows. Section 2 introduces variational preferences. Section 3 recalls and characterizes for MMR agents, the concepts of useful trading directions, of no-arbitrage prices and of collective absence of arbitrage. The no half-line condition is introduced. Section 4 deals with existence of efficient allocations and equilibria. Appendix A provides all the proofs that are not given in the main part of the paper.

2. The model and variational preferences

We consider a standard Arrow–Debreu model of complete contingent security markets. There are two dates, 0 and 1. At date 0, there is uncertainty about which state \( s \) from a state space \( \Omega = \{1, \ldots, k\} \) will occur at date 1. At date 0, agents who are uncertain about their future endowments trade contingent claims for date 1. The space of contingent claims is the set of random variables from \( \Omega \to \mathbb{R} \). The random variable \( X \) which equals \( x_1 \) in state 1, \( x_2 \) in state 2 and \( x_k \) in state \( k \), is identified with the vector in \( X \in \mathbb{R}^k \), \( X = (x_1, \ldots, x_k) \). Let \( \Delta = \{\pi \in \mathbb{R}_+^k : \sum_{i=1}^k \pi_i = 1\} \) be the probability simplex in \( \mathbb{R}^k \). For a given \( \pi \in \Delta \), we denote by \( E_\pi(X) := \sum_{i=1}^k \pi_i x_i \) the expectation of \( X \). Two probabilities \( \pi \) and \( \pi \) such that \( \pi \) is absolutely continuous with respect to \( q \) are denoted by \( \pi \ll q \), two equivalent probabilities \( p \) and \( \pi \) will be denoted by \( p \simeq \pi \).

For \( \pi \in P \), \( I_\pi = \{s \mid \pi_s > 0\} \). We denote by int \( \Delta = \{p \in \Delta \mid p_s > 0 \text{ for all } s\} \) and for \( A \subseteq \Delta \), int \( A = \{p \in A \mid \exists \text{ a ball } B(p, \epsilon) \text{ s.t. } B(p, \epsilon) \cap \text{int } \Delta \subseteq A\} \). Finally, for a given price \( p \in \mathbb{R}^k \), \( p : X := \sum_{i=1}^k p_i x_i \), the price of \( X \).

There are \( m \) agents indexed by \( i = 1, \ldots, m \). Agent \( i \) has an endowment \( E^i \in \mathbb{R}^k \) of contingent claims. We denote by \( (E^i)^m_{i=1} \) the \( m \)-tuple of endowments and by \( E = \sum_{i=1}^m E^i \) aggregate endowment. We assume that each agent has a preference order \( \succeq \) over \( \mathbb{R}^k \) represented by a utility function \( V \) which verify: there exist a concave, strictly increasing differentiable utility index \( u : \mathbb{R} \to \mathbb{R} \) and a convex lower semi-continuous function \( c : \Delta \to [0, \infty] \) such that the utility \( V : \mathbb{R}^k \to \mathbb{R} \) is given by

\[
V(X) = \min_{\pi \in \Delta} E_\pi(u(X)) + c(\pi).
\]

Utilities of type (1) have been axiomatized by Maccheroni, Marinacci and Rustichini [23] on an Anscombe–Aumann domain and capture risk and uncertainty. Monetary payoffs are just considered here. Risk aversion is modelled by \( u \) being concave and \( V_1 \) is more risk averse than \( V_2 \) if \( u_1 \) is more risk averse than \( u_2 \). From Arrow–Pratt’s theorem, \( u_1 \) is more risk averse than \( u_2 \) if and only if \( u_1 = \psi \circ u_2 \) for some \( \psi \) concave increasing. According to Maccheroni et al. [23], \( \succeq_1 \) is more ambiguity averse than \( \succeq_2 \) if and only if \( u_1 = au_2 + b \) for some \( a > 0, b \in \mathbb{R} \) and \( c_1 \leq c_2 \) provided \( u_1 = u_2 \). Hence \( c \) is an index of ambiguity aversion.

Variational preferences nest many of the models developed to study ambiguity in the decision theoretic, financial and economic literatures, in particular:

- the maxmin expected utility of Gilboa and Schmeidler [13]
  \[
  V(X) = \min_{\pi \in P} E_\pi(u(X))
  \]
  which is obtained for \( c = \delta_p \), an indicator function of a convex compact subset \( P \) of \( \Delta \) \( (c(\pi) = 0 \text{ if } \pi \in P \text{ and } c(\pi) = \infty \text{ otherwise}) \).
- the multiplier utility used by Hansen and Sargent [17] where
  \[
c(\pi \mid p) = \begin{cases} 
  \theta \sum_{\pi_s} \pi_s \log \frac{\pi_s}{p_s} & \text{if } \pi \ll p, \\
  \infty & \text{otherwise},
\end{cases}
\]
θ > 0 is a parameter of ambiguity aversion and the cost function \( \pi \to \sum_s \pi_s \log \frac{\pi_s}{p_s} \) is the relative entropy between the probabilities \( \pi \) and \( p \),

- the monetary utility functions which fulfill (1) with \( u(x) = x \). The opposite of a monetary utility function is a convex measure of risk. Monetary utilities with cost function \( c = \delta_P \), the indicator function of a convex compact subset \( P \) of \( \Delta_1 \), \( V(X) = \min_{\pi \in P} E_\pi(X) \) correspond to coherent measures of risk.

Let \( P = \text{dom } c = \{ \pi : c(\pi) < +\infty \} \) be the set of effective priors. Then

\[
V(X) = \min_P E_\pi(u(X)) + c(\pi). \tag{3}
\]

For a fixed \( u \), the more ambiguity aversion, the smaller \( c \) and the larger is \( P \).

3. Useful trades, no-arbitrage and no-half lines

In this section, we recall and characterize the concepts of useful trading directions, no-arbitrage prices and collective absence of arbitrage for a utility of type (3). We then define the no-half line condition.

3.1. Useful vectors

Let \( V \) be a utility of type (3). For \( X \in \mathbb{R}^k \), let \( \hat{P}(X) = \{ Y \in \mathbb{R}^k \mid V(Y) \geq V(X) \} \) be the set of contingent claims preferred to \( X \) and let \( R(X) \) be its asymptotic cone (see Rockafellar [31, Section 8]). Since \( V \) is concave, by Rockafellar’s Theorem 8.7 in [31], \( R(X) \) is independent of \( X \) and called the set of useful vectors for \( V \). It will be denoted by \( R \). We recall that

\[
R = \{ W \in \mathbb{R}^k \mid V(\lambda W) \geq V(0), \text{ for all } \lambda \geq 0 \}.
\]

We first show that a utility \( V \) of type (3) is the minimum of a family of affine combinations of linear expectations over a set of priors \( \hat{P} \) which, in the financial tradition, we call the risk adjusted set of priors. Indeed, since \( u \) is concave and differentiable,

\[
u(x) = \min_{z \in \mathbb{R}} \left\{ u'(z) x + u(z) - u'(z) z \right\}. \tag{4}\]

We may therefore characterize \( V \) as follows.

**Lemma 1.** Let \( V \) fulfill (3) and \( u \) be nonlinear. For any \( X \in \mathbb{R}^k \), we have

\[
V(X) = \min_{\eta} \left\{ \left( E_{\pi} u'(Z) \right) \left\{ \sum_s \frac{\pi_s u'(z_s)}{E_{\pi} u'(Z)} x_s + \frac{\gamma(\eta)}{E_{\pi} u'(Z)} \right\} \right\} \tag{5}
\]

where \( \eta = (\pi, Z) \in P \times \mathbb{R}^k \) and \( \gamma(\eta) = E_{\pi} u(Z) - E_{\pi} (u'(Z) Z) + c(\pi) \).

The above representation leads us to introduce

\[
\hat{P} = \left\{ p \in \Delta \mid \exists \pi \in P, Z \in \mathbb{R}^k \text{ s.t. } p_s = \frac{\pi_s u'(z_s)}{E_{\pi} u'(Z)}, \forall s = 1, \ldots, k \right\}. \tag{6}
\]

Next proposition gathers some properties of \( \hat{P} \). We now on use the following notations. Let \( a = u'(+\infty) \) and \( b = u'(-\infty) \) be the asymptotic slopes of \( u \) and \( t = \frac{a}{b} \) be their ratio. Note that \( t = 0 \) if and only if \( a = 0 \) or \( b = +\infty \) while \( t = 1 \) if and only if \( u \) is affine. It follows from
Arrow–Pratt’s theorem that $t$ is a measure of risk tolerance for an expected utility maximizer: the more risk averse the agent and the smaller is $t$.

**Proposition 1.**

1. $P \subseteq \tilde{P}$. $\tilde{P} = P$ when $t = 1$ or $P = \text{int} \Delta$ or $P = \Delta$.
2. The set $\tilde{P}$ is convex.
3. If $t = 0$, $\tilde{P} = \{ p \in \Delta \mid \exists \pi \in P, \pi \simeq p \}$. If moreover, $P \cap \text{int} \Delta \neq \emptyset$, then $\text{int} \Delta \subseteq \tilde{P}$.
4. The more ambiguous or the more risk averse the agent and the larger is $\tilde{P}$.

We may now characterize the useful vectors of a utility of type (3).

**Proposition 2.** Let $V$ fulfill (3) with $t \leq 1$. Then

1. $R = \{ W \in \mathbb{R}^k \mid E_p(W) \geq 0, \text{ for all } p \in \tilde{P} \}$.
2. If $t = 0$ and $P \cap \text{int} \Delta \neq \emptyset$ or if $\tilde{P} = \Delta$, then $R = \mathbb{R}^k_+$.

It follows from Propositions 1 and 2 that for a utility of type (3) the more ambiguous or the more risk averse the agent, the larger is $\tilde{P}$, the smaller are the sets of useful vectors.

**Remark 1.** One can show (see Appendix A) that the condition

$$E_p(W) \geq 0, \text{ for all } p \in \tilde{P}$$

is equivalent to $t E_{\pi}(W_+) - E_{\pi}(W_-) \geq 0, \text{ for all } \pi \in P$,

where $(W_+)_l = w_l$ if $x_l \geq 0$ and $(W_-)_l = -w_l$ if $w_l \leq 0$. When $P$ is a singleton, (8) is the incomplete mean condition given by Bertsekas [3] and Hart [18].

### 3.2. No-arbitrage prices

The second concept that we recall is that of a no-arbitrage price, a price for which no agent can make costless unbounded utility nondecreasing purchases.

**Definition 1.** A price vector $p \in \mathbb{R}^k$ is a “no-arbitrage price” for agent $i$ if $p \cdot W > 0$, for all $W \in \mathbb{R}^i \setminus \{0\}$. A price vector $p \in \mathbb{R}^k$ is a “no-arbitrage price” for the economy if it is a no-arbitrage price for each agent.

For $A \subseteq \mathbb{R}^d$, we denote $A^0$ the polar of $A$ where we recall that $A^0 = \{ p \in \mathbb{R}^d \mid p \cdot A \leq 0, \text{ for all } X \in A \}$.

Let $S^i$ denote the set of no-arbitrage prices for $i$. Then $S^i = -\text{int}(R^i)^0$. A price vector $p \in \mathbb{R}^k$ is a “no-arbitrage price” for the economy if and only if $p \in \bigcap_i S^i = -\bigcap_i \text{int}(R^i)^0$. From Proposition 2, we may characterize the set of no-arbitrage prices for agent $i$ and for the economy. A normalized no-arbitrage price for an agent $i$ with $t^i < 1$ is a strictly positive risk adjusted probability in $\tilde{P}^i$ that fulfills (9) below and for an agent $i$ with $t^i = 1$, a strictly positive probability in $\text{int} P^i$.
Proposition 3. Let $V^i$ fulfill (3) for each $i$. Then

1. the set of no-arbitrage prices for agent $i$ is $S^i = \text{cone int } \tilde{P}^i$.
2. If $t^i < 1$, $p \in \text{int } \tilde{P}^i$ if and only if
   \[
   \exists \pi \in P^i \cap \text{int } \Delta, \ Z \in \mathbb{R}^k, \ \forall s, \ a < u'(z_s) < b \quad \text{and} \quad p_s = \frac{\pi_s u'(z_s)}{E\pi u'(Z)}.
   \] (9)

   Hence $S^i \neq \emptyset$ if and only if $P^i \cap \text{int } \Delta \neq \emptyset$.

   If $t^i = 1$, then $S^i \neq \emptyset$ if and only if, $\text{int } P^i \neq \emptyset$.

3. The set of no-arbitrage prices for the economy is
   \[
   \bigcap_i S^i = \text{cone } \bigcap_i \text{int } \tilde{P}^i.
   \]

   Let $I_1 = \{i \mid t^i < 1\}$ and $I_2 = \{i \mid t^i = 1\}$. Then $\bigcap_i S^i \neq \emptyset$ if and only if, for any $i \in I_1$, there exist $\pi^i \in P^i \cap \text{int } \Delta, \ Z^i \in \mathbb{R}^k$ with $u^i(+\infty) < u^i(z_s^i) < u^i(-\infty)$ for all $s$ and $\pi \in \bigcap_{i \in I_2} \text{int } P^i$ such that, for all $i \in I_1, j \in I_1, s = 1, \ldots, k$,
   \[
   \frac{\pi^i_s u^i(z_s^i)}{E\pi_s u^i(Z^i)} = \frac{\pi^j_s u^j(z_s^j)}{E\pi_j u^j(Z^j)} = \pi_s.
   \]

   In order to insure existence of no-arbitrage price for agent $i$, we shall now on assume that if $t^i < 1$, $P^i \cap \text{int } \Delta \neq \emptyset$ and if $t^i = 1$, $\text{int } P^i \neq \emptyset$.

Let us give simple sufficient conditions for the non-emptiness of the set of no-arbitrage prices for the economy. The first condition is that agents have an “open” set of priors in common, the second is that they are infinitely risk averse.

Corollary 1.

1. If $\bigcap_i \text{int } P^i \neq \emptyset$, then $\bigcap_i S^i \neq \emptyset$.
2. If $t^i = 0$ and $P^i \cap \text{int } \Delta \neq \emptyset$ for all $i$, then $\bigcap_i S^i = \text{int } \mathbb{R}^k$.

3.3. Collective absence of arbitrage

From now on, a feasible trade is an $m$-tuple $W^1, \ldots, W^m$ with $W^i \in \mathbb{R}^k$ for all $i$ and $\sum_i W^i = 0$. We recall the no-unbounded-arbitrage condition (NUBA) introduced by Page [26] which requires inexistence of unbounded utility nondecreasing feasible trades.

Definition 2. The economy satisfies the NUBA condition if $\sum_i W^i = 0$ and $W^i \in R^i$ for all $i$, implies $W^i = 0$ for all $i$.

From Proposition 2, we may now characterize the NUBA condition.

Corollary 2. NUBA is equivalent to: there exists no feasible trade $W^1, \ldots, W^m$ with $W^i \neq 0$ for some $i$ that fulfills $E\pi(W^i) \geq 0$ for all $i$ and $\pi \in \tilde{P}^i$.

3.4. The no-half line condition

Definition 3. A trade $W \in \mathbb{R}^k \setminus \{0\}$ is a half-line if there exists $X \in \mathbb{R}^k$ such that $V(X + \lambda W) = V(X)$ for all $\lambda \geq 0$. 
The next lemma characterizes then no-half line condition in the case of a risk averse expected utility maximizer and of a risk neutral MMR agent. Sufficient conditions are provided for a risk averse MMR utility to have no-half lines as well as necessary conditions. For a given \( X \in \mathbb{R}^k \), let
\[
P(X) = \{ \pi \in P \mid V(X) = E(\pi(u(X)) + c(\pi) \}
\]
be the set of minimizing probabilities at \( X \).

**Lemma 2.**

1. Let \( V \) fulfill (3) and \( t = 1 \). Then \( V \) has no-half line if and only if \( P(X) \subseteq \text{int} \, P \) for any \( X \in \mathbb{R}^k \).
2. Let \( V \) fulfill (3). Assume that \( P(X) \subseteq \text{int} \Delta \) for any \( X \in \mathbb{R}^k \) and that \( a < u'(x) \) for all \( x \) or \( u'(x) < b \) for all \( x \) (no risk neutrality at infinity). Then \( V \) has no-half line.
3. If \( V \) has no-half line, then \( P(X) \subseteq \text{int} \Delta \) for any \( X \in \mathbb{R}^k \). If \( V \) fulfills (2) and has no-half line, then \( P \subseteq \text{int} \Delta \).
4. Let \( V(X) = E_\pi(u(X)) \). Then \( V \) has no-half line if and only if \( \pi \in \Delta \) and \( a < u'(x) \) for all \( x \) or \( u'(x) < b \) for all \( x \).

When \( t = 1 \), the no half-line condition is fulfilled for example in the case of entropy but it is not fulfilled for utilities of the type \( V(X) = \min_{\pi \in P} E_\pi(X) \). \( P \) convex compact since the minimizing probabilities are at the boundary of \( \pi \). When \( t < 1 \), the no-half line condition is fulfilled for Gilboa–Schmeidler’s utilities, \( V(X) = \min_{\pi \in P} E(u(X)) \) if \( P \subseteq \text{int} \Delta \) and if the agent is not risk neutral at infinity. Strictly concave utilities have no half-lines. The strict concavity of \( V \) is characterized in Lemma 4 in Appendix A.

4. Existence of efficient allocations and equilibria

4.1. Concepts in equilibrium theory

Let us recall standard concepts in equilibrium theory.

Given \((E^i)_{i=1}^m\), an allocation \((X^i)_{i=1}^m \in (\mathbb{R}^k)^m\) is attainable if \( \sum_{i=1}^m X^i = E \). The set of individually rational attainable allocations \(A((E^i)_{i=1}^m)\) is defined by
\[
A((E^i)_{i=1}^m) = \left\{ (X^i)_{i=1}^m \in (\mathbb{R}^k)^m \mid \sum_{i=1}^m X^i = E \text{ and } V^i(X^i) \geq V^i(E^i), \forall i \right\}.
\]
The individually rational utility set \( U((E^i)_{i=1}^m) \) is defined by
\[
U((E^i)_{i=1}^m) = \{ v \in \mathbb{R}^m \mid \exists X \in A((E^i)_{i=1}^m) \text{ s.t. } V^i(E^i) \leq v_i \leq V^i(X^i), \forall i \}.
\]

**Definition 4.** Given \((E^i)_{i=1}^m\), an attainable allocation \((X^i)_{i=1}^m \) is efficient if there does not exist \((X'^i)_{i=1}^m \) attainable such that \( V_i(X'_i) \geq V_i(X_i) \) for all \( i \) with a strict inequality for some \( i \). It is individually rational efficient if it is efficient and \( V^i(X'^i) \geq V^i(E^i) \) for all \( i \).

**Definition 5.** A pair \((X^*, p^*) \in A((E^i)_{i=1}^m) \times \mathbb{R}^k \setminus \{0\} \) is a contingent Arrow–Debreu equilibrium if
1. for each agent \( i \) and \( X^i \in \mathbb{R}^k \), \( V^i(X^i) > V(X^i) \) implies \( p^* \cdot X^i > p^* \cdot X^i \),
2. for each agent \( i \), \( p^* \cdot X^i = p^* \cdot E^i \).
4.2. Necessary and sufficient conditions

We first characterize the existence of efficient allocations and of equilibria under the condition that the utilities do not contain half-lines. They follow from Theorem 1 in Appendix A and Propositions 2 and 3.

**Proposition 4.** Let \( V^i \) fulfill (3). Then the following assertions are equivalent:

1. \( \bigcap_i \text{int } \tilde{P}^i \neq \emptyset \) with \( \tilde{P}^i = P^i \) for any \( i \in I_2 \).
2. For any \( i \in I_1 \), there exist \( \pi^i \in P^i \cap \text{int } \Delta \), and \( Z^i \in \mathbb{R}^k \) with \( a^i < u_i'(z^i_s) < b^i \) for all \( s \) and \( \pi \in \bigcap_{i \in I_2} \text{int } P^i \) such that for all \( i \in I_1, j \in I_1, s = 1, \ldots, k \), we have
   \[
   \frac{\pi^i_s u_i'(z^i_s)}{E_{\pi^i} u_i'(Z^i)} = \frac{\pi^j_s u_j'(z^j_s)}{E_{\pi^j} u_j'(Z^j)} = \pi^s.
   \]
3. There exists no feasible trade \( W^1, \ldots, W^m \) with \( W^i \neq 0 \) for some \( i \) and \( E_{\pi}(W^i) \geq 0 \) for all \( \pi \in \tilde{P}^i \) and for all \( i \).

Any of the above assertions implies any of the following assertions:

4. There exists an individually rational efficient allocation for any distribution of initial endowments.
5. There exists an equilibrium for any distribution of initial endowments.

If furthermore \( V^i \) has no half-line for all \( i \), all these assertions are equivalent and any equilibrium price is a no-arbitrage price.

One can apply Proposition 4 to provide sufficient conditions for existence of efficient allocations in terms of priors and utility indices.

**Corollary 3.** Assume that \( V^i \) fulfills (3) for all \( i \) and that \( \bigcap_i \text{int } P^i \neq \emptyset \). Then there exist efficient allocations.

In the previous corollary, we assumed existence of a common prior. The next proposition shows that if agents are all infinitely risk averse a common prior is not necessary for existence of efficient allocations.

**Proposition 5.** Assume that \( V^i \) fulfills (3) with \( t^i = 0 \) for all \( i \). Assume that \( P^i \cap \text{int } \Delta \neq \emptyset \) for all \( i \). Then \( \bigcap_i \text{int } \tilde{P}^i = \text{int } \Delta \) and there exist equilibria for any distribution of endowments \( (E^i)_{i=1}^m \).

The previous proposition applies in particular to the case of agents with single heterogeneous beliefs.

**Corollary 4.** Let \( V^i = E_{\pi^i} (u^i(X)) \) with \( t^i = 0, \pi^i \in \text{int } \Delta \) for all \( i \) and \( \pi^i \neq \pi^j \) for all \( (i, j) \). Then \( \bigcap_i \text{int } \tilde{P}^i = \text{int } \Delta \) and there exist equilibria for any distribution of endowments \( (E^i)_{i=1}^m \).
4.3. Necessary conditions for existence of efficient allocations

The previous subsection provides sufficient conditions for existence of equilibria which are necessary if utilities have no half-lines. Necessary conditions are now given without assuming the no half-line conditions.

**Proposition 6.** Let $V^i$ fulfill (3) for each $i$. If there exists an efficient allocation for some distributions of endowments $(E^i)_{i=1}^m$, then

1. $\bigcap_i \tilde{P}^i \neq \emptyset$,
2. there exists no feasible trade $W^1, \ldots, W^n$ fulfilling $E_\pi(W^i) > 0$, $\forall \pi \in \tilde{P}^i$,
3. for any distribution of endowments $(E^i)_{i=1}^m$, the individually rational utility set $U((E^i)_{i=1}^m)$ is bounded.

4.4. Link with the literature

We now compare our existence results for markets with short-selling and the standard characterization of efficient allocations for given aggregate endowments often given without referring to efficiency with short-selling. Let $P$ be a set of prior, $X \in \mathbb{R}^k$ and let

$$\tilde{P}(X) = \left\{ p \in \Delta \mid \exists \pi \in P(X), \text{ s.t. } p_s = \frac{\pi_s u'(x_s)}{E_\pi u'(X)}, \forall s = 1, \ldots, k \right\}$$

(10)

be the set of risk adjusted probabilities of the minimizing probabilities at $X$.

**Proposition 7.** Let $V^i$ fulfill (3) for all $i$ be given. The allocation $(\bar{X}^i)_{i=1}^m$ is efficient for some aggregate endowment $E \in \mathbb{R}^k$ if any of the equivalent following conditions is fulfilled:

1. $\bigcap_i \tilde{P}^i(\bar{X}^i) \neq \emptyset$,
2. there exists no feasible trade $(W^i)_{i=1}^m$ such that $E_\pi W^i > 0$ for all $\pi \in \tilde{P}^i(\bar{X}^i)$ and all $i$.

Let us first remark that assertion 1 of Proposition 6 follows from assertion 1 above. Let us next compare assertion 1 of Proposition 7 and assertion 2 of Proposition 4. In Proposition 7, the common risk adjusted probability is minimizing at $\bar{X}^i$ for each $i$. If $V^i$ has no half-line and if $t = 1$, from Lemma 2 assertion 1, the minimizing probabilities must be in int $P$ while if $t < 1$, from assertion 3, they must be in int $\Delta$. In assertion 2 of Proposition 4, $\pi^i \in \text{int } P$ if $t = 1$ and $\pi^i \in \text{int } \Delta$ if $t < 1$ but $\pi^i$ need not be a minimizing probability at $Z^i$. Hence if agents utilities have no half line, the condition $\bigcap_i \tilde{P}^i(\bar{X}^i) \neq \emptyset$ implies that the intersections of the interiors of the risk adjusted priors is non-empty and the existence of an allocation for any distribution of endowments. When utilities have half-lines, assertion 2 of Proposition 4 and assertion 1 of Proposition 7 are both sufficient conditions but assertion 1 of Proposition 7 depends on $(\bar{X}^i)_{i=1}^m$ which is unknown.

The conditions of Proposition 7 also characterize efficiency of interior allocations in consumption models and have been used already in a large number of papers including Billot et al. [4], Dana [5,6], Epstein and Wang [10], Kajii and Ui [22], Rigotti and Shannon [29] and Rigotti et al. [30].

The equivalence between the two conditions of Proposition 7 is also proven in papers on the no-trade, e.g. Samet [32], Kajii and Ui [21].
4.5. A final remark

In order to state Proposition 4 and insure existence of no-arbitrage prices for agent $i$, we had to assume that if $t^i_1 < 1$, $P^i \cap \text{int} \Delta \neq \emptyset$. This means that agent $i$ has at least one prior that gives positive weight to each state of the world. When this condition is not satisfied, since $P^i_1$ is convex, there are states of the world that agent $i$ considers as totally unlikely: equivalently

$$\{s \mid \pi_s = 0, \text{ for all } \pi \in P\} \neq \emptyset$$

indeed, assume on the contrary that for any $s$, there exists $\pi^i(s) \in P^i$ such that $\pi^i(s) > 0$. Let $\lambda \in \text{int} \Delta$. Then $\nu^i = \sum \lambda s \pi^i(s) \in P^i \cap \text{int} \Delta$, contradicting our assumption.

The theory of existence of equilibrium has been refined by Allouch et al. [1] and it follows from their paper that a no-arbitrage condition is that agents agree on which states of the world are irrelevant for them. When this condition is fulfilled, one is brought down to consider a smaller state space where the theory developed in this paper can be applied. The interested reader is referred to Dana and Le Van’s working paper [9].

Appendix A. Proofs

A.1. Proof of Lemma 1

For any $\eta = (\pi, Z)$, let $\gamma(\eta) = E_\pi u(Z) - E_\pi (u'(Z)Z) + c(\pi)$. We then have

$$(E_\pi u'(Z)) \left\{ \sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} x_s + \frac{\gamma(\eta)}{E_\pi u'(Z)} \right\} = \sum_s \pi_s u'(z_s) x_s + \gamma(\pi, Z)$$

$$\geq \min_{\pi} \left\{ \min_Z \left\{ \sum_s \pi_s u'(z_s) x_s + \gamma(\pi, Z) \right\} \right\}$$

$$= \min_{\pi} \left\{ \sum_s \pi_s u(x_s) + c(\pi) \right\} = V(X)$$

where the last equality follows from (4). Hence

$$\min_{\eta} \left\{ (E_\pi u'(Z)) \left\{ \sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} x_s + \frac{\gamma(\eta)}{E_\pi u'(Z)} \right\} \right\} \geq V(X).$$

Conversely, by definition of $\gamma$, we have for any $\pi \in P$

$$\sum_s \pi_s u(x_s) + c(\pi) = \sum_s \pi_s u'(x_s) x_s + \gamma(\pi, X)$$

$$\geq \min_{\pi, Z} \sum_s \pi_s u'(z_s) x_s + \gamma(\pi, Z)$$

hence,

$$V(X) \geq \min_{\pi, Z} \sum_s \pi_s u'(z_s) x_s + \gamma(\pi, Z)$$

$$= \min_{\eta} \left\{ (E_\pi u'(Z)) \left\{ \sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} x_s + \frac{\gamma(\eta)}{E_\pi u'(Z)} \right\} \right\}$$

proving Lemma 1.
A.2. Proof of Proposition 1

To prove that $P \subseteq \tilde{P}$, it suffices to take $Z$ constant in (6).

To prove assertion 2, let

$$
\tilde{Q} = \left\{ \lambda (\pi_s u'(z_s))_s ; \lambda > 0, \pi \in P, Z \in \mathbb{R}^k \right\}
$$

be the cone generated by $P$. Since $\tilde{P} = \tilde{Q} \cap \Delta$, it suffices to prove that $\tilde{Q}$ is convex. To this end, let $\lambda (\pi_s u'(z_s))_s \in \tilde{Q}$, $\lambda' (\pi_s' u'(z_s'))_s \in \tilde{Q}$ and $\alpha \in (0, 1)$. Then for any $s$,

$$
\begin{align*}
(\alpha \lambda \pi_s + (1 - \alpha) \lambda' \pi'_s)u'(+\infty) &\leq \alpha \lambda \pi_s u'(z_s) + (1 - \alpha) \lambda' \pi'_s u'(z_s') \\
&\leq (\alpha \lambda \pi_s + (1 - \alpha) \lambda' \pi'_s)u'(-\infty).
\end{align*}
$$

Hence, there exists $\zeta_s$ which satisfies

$$
(\alpha \lambda \pi_s + (1 - \alpha) \lambda' \pi'_s)u'((\zeta_s)) = \alpha \lambda \pi_s u'(z_s) + (1 - \alpha) \lambda' \pi'_s u'(z_s').
$$

Define $\nu = \frac{\alpha \lambda + (1 - \alpha) \lambda'}{\alpha \lambda + (1 - \alpha) \lambda'}$. Then, $\nu \in P$ and for any $s$,

$$
\begin{align*}
\alpha \lambda \pi_s u'(z_s) + (1 - \alpha) \lambda' \pi'_s u'(z_s') = (\alpha \lambda + (1 - \alpha) \lambda') \nu u'(\zeta_s),
\end{align*}
$$

proving the convexity of $Q$.

To prove assertion 3, from its definition $\tilde{P} \subseteq \{ p \in \Delta \mid \exists \pi \in P, \pi \simeq p \}$. If $u'(\infty) = 0$ or $u'(-\infty) = +\infty$, then for any $\pi \simeq \pi \in P$, there exists $\lambda > 0$ such that

$$
u'((\infty)) < \lambda \frac{p_s}{\pi_s} < \nu'((-\infty))
$$

and thus, there exists $Z \in \mathbb{R}^k$ such that, for all $s \in I_\pi$, $\frac{p_s}{\pi_s} = \frac{u'(z_s)}{E_\pi u'(Z)}$, hence $\tilde{P} = \{ p \in \Delta \mid \exists \pi \in P, \pi \simeq p \}$. If there exists $\pi \in P \cap \text{int} \Delta$, then $\tilde{P}$ contains $\text{int} \Delta$ the set of strictly positive probabilities which are equivalent to $\pi$.

To prove the last assertion, clearly the more ambiguous the agent, the larger is $P$ and hence the larger is $\tilde{P}$. Let us show that the more risk averse the agent, the larger is $\tilde{P}$. Indeed, if $v$ is more risk averse than $u$, then from Arrow–Pratt’s theorem, $v = \psi \circ u$ with $\psi$ concave. Let $\tilde{P}_u$ and $\tilde{P}_v$ be the sets of risk adjusted priors associated to $u$ and $v$. Assume that $p \in \tilde{P}_u$. Then there exist $\pi$ and $Z \in \mathbb{R}^k$ such that, for all $s \in I_\pi$,

$$
u'(\infty) \leq \frac{p_s}{\pi_s} E_\pi u'(Z) \leq \nu'((-\infty)).
$$

(11)

If $\nu'(\infty) = 0$ or $\nu'((-\infty)) = \infty$, then from assertion 2, $\tilde{P}_v = \{ p \in \Delta \mid \exists \pi \in P, \pi \simeq p \}$, hence $p \in \tilde{P}_v$. Let us therefore assume that $0 < \nu'(\infty) < \nu'((-\infty)) < \infty$. We first obtain from (11) that

$$
1 \leq \frac{p_s}{\pi_s} \frac{E_\pi u'(Z)}{u'(+\infty)} \leq u'((-\infty)),
$$

Since $0 < \nu'(\infty)$ and $\nu'(\infty) = \nu'(\infty)\psi(u(+\infty))$, we have $u'(\infty) > 0$ and $\psi u'(+\infty) > 0$, therefore

$$
\begin{align*}
\frac{\nu'(-\infty)}{\nu'(\infty)} = \frac{\psi'(u(-\infty))}{\psi'(u(+\infty))} \leq \frac{\psi'(u(-\infty))}{\psi'(u(+\infty))} < \frac{u'(-\infty)}{u'(\infty)}
\end{align*}
$$

since $\psi$ is concave but not linear on $u(\mathbb{R})$. Hence

$$
1 \leq \frac{p_s}{\pi_s} \frac{E_\pi u'(Z)}{u'(+\infty)} < \frac{\nu'(-\infty)}{\nu'(\infty)}.
$$
Let $\lambda = \frac{v'(\infty)E_\pi u'(Z)}{u'(\infty)}$, we obtain that

$$v'(\infty) < \lambda \frac{p_s}{\pi_s} < v'(-\infty)$$

and thus, there exists $Z' \in \mathbb{R}^k$ such that, for all $s \in I_\pi$, $\frac{p_s}{\pi_s} = \frac{v'(z'_s)}{E_\pi v'(Z)}$ which proves that $p \in \tilde{P}_\nu$.

A.3. Proof of Proposition 2

We first prove assertion 1. From (5), if $W$ is a useful vector, then for any $(\pi, Z)$ and any $\lambda > 0$, we have:

$$\left(\frac{E_\pi u'(Z)}{E_\pi u'(Z)}\right)\left\{\sum_s \left(\frac{\pi_s u'(z_s)}{E_\pi u'(Z)}(\lambda w_s) + \gamma(\pi, Z)\right)\right\} \geq V(0).$$

Dividing by $\lambda$ and letting $\lambda$ go to $+\infty$, we obtain:

$$\sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} w_s \geq 0, \quad \forall (\pi, Z).$$

Conversely, if $\sum_s \frac{\pi_s u'(z_s)}{E_\pi u'(Z)} w_s \geq 0, \forall (\pi, Z)$, then for any $\lambda > 0$,

$$\left(\frac{E_\pi u'(Z)}{E_\pi u'(Z)}\right)\left\{\sum_s \left(\frac{\lambda w_s \pi_s u'(z_s)}{E_\pi u'(Z)} + \gamma(\pi, Z)\right)\right\} \geq \gamma(\pi, Z)$$

and hence $V(\lambda W) \geq V(0), \forall \lambda \geq 0$ and $W$ is useful. Assertion 2 follows from assertion 3 of Proposition 1.

A.4. Proof of Remark 1

(7) is equivalent to $\sum_l \pi_l u'(x_l)w_l \geq 0$ for every $X \in \mathbb{R}^k$ and $\pi \in P$. Letting $x_l$ go to $+\infty$ for any $l$ such that $w_l > 0$ and $x_l$ go to $-\infty$ for all such that $w_l < 0$ and dividing by $b$, we obtain (8). Conversely, since, for any $X \in \mathbb{R}^k$ and $\pi \in P$,

$$\frac{b}{E(u'(X))} \sum_l \pi_l u'(x_l)w_l \geq \frac{b}{E(u'(X))} \left( \sum_{l|w_l \geq 0} \pi_l w_l + \sum_{l|w_l < 0} \pi_l w_l \right)$$

(8) implies (7).

A.5. Proof of Proposition 3

In order to determine no-arbitrage prices and prove Proposition 3, we need to characterize $\text{int} \tilde{P}$. In the case $t = 1$, we have $\text{int} \tilde{P} = \text{int} P$. In the next lemma, we characterize $\text{int} \tilde{P}$ in the case $t < 1$.

Lemma 3. Let $V$ fulfill (3) with $t < 1$. Then $p \in \text{int} \tilde{P}$ if and only if it satisfies

$$\exists \pi \in P \cap \text{int} \Delta, \ Z \in \mathbb{R}^k, \ s.t. \ \forall s, \ a < u'(z_s) < b, \ and \ p_s = \frac{\pi_s u'(z_s)}{E_\pi u'(Z)}.$$

(9)
Proof. Let us first show that if \( p \) satisfies (9), then \( p \in \text{int} \tilde{P} \). Indeed, we have \( p_\pi u'(z_s) \) for any \( s \). For any \( \epsilon \in \mathbb{R} \) close to 0, we can find \( z'_s \) such that \( \frac{p_{\pi} u'(z'_s)}{\pi s(Z)} < b \). Indeed, since \( a < \frac{(p_{\pi} + \epsilon) u'(Z)}{\pi s(1 + k \epsilon)} \) \( b \), for \( \epsilon \) small enough, we have \( a < \frac{(p_{\pi} + \epsilon) u'(Z)}{\pi s(1 + k \epsilon)} \). Thus there exists \( z'_s \) such that \( \frac{(p_{\pi} + \epsilon) u'(Z)}{\pi s(1 + k \epsilon)} = u'(z'_s) \). We then have \( \pi s u'(Z) = \pi s u'(Z') \) which implies that \( \frac{p_{\pi} u'(z'_s)}{\pi s(Z)} = \pi s u'(Z') \).

In other words, there exist an open set containing \( p \) which is included in \( \tilde{P} \). Hence, \( p \in \text{int} \tilde{P} \).

Since \( \tilde{P} \) is convex, to prove the converse, from Rockafellar’s Theorem 6.4, when \( \text{int} \tilde{P} \neq \emptyset \), \( p \in \text{int} \tilde{P} \) if and only if, for every \( p' \in \tilde{P} \), there exists \( p'' \in \tilde{P} \) such that \( p = \alpha p'' + (1 - \alpha) p' \) with \( \alpha \in [0, 1] \). Consider a \( p' \) that verifies (9). Let \( \lambda' = \frac{1}{\pi s u'(Z')} \) and \( \lambda'' = \frac{1}{\pi s u'(Z'')} \). From the proof of Proposition 1 assertion 2, we have that \( p_s = \frac{u'(z_s)}{\pi s u'(Z)} \) with

\[
\pi s = \frac{\alpha \lambda'' \pi' s + (1 - \alpha) \lambda' \pi s'}{\alpha \lambda'' + (1 - \alpha) \lambda'} ,
\]

\[
a < u'(z_s) = \frac{\alpha \lambda'' \pi' u'(z_s') + (1 - \alpha) \lambda' \pi s' u'(z_s')}{\alpha \lambda'' \pi s' + (1 - \alpha) \lambda'} < b.
\]

Since \( \pi' \in P \cap \Delta \), \( \pi \in P \cap \text{int} \Delta \). Hence (9) is fulfilled. \( \square \)

Let us now prove Proposition 3. Given a subset \( A \), let \( \text{cl} A \) be its closure.

To prove assertion 1, from Proposition 2, \( R^i = \{ W \in \mathbb{R}^k \mid E_\pi (W) \geq 0 \} \), for all \( \pi \in \tilde{P}^i \). Hence \((R^i)^0 \) is the closed cone generated by \( \tilde{P}^i \). Since \( \tilde{P}^i \) is convex, \( S^i = \text{int cl cone} \tilde{P}^i \). Since cone \( \tilde{P}^i \) is convex, \( \text{int cl cone} \tilde{P}^i = \text{cone int} \tilde{P}^i \).

The first part of assertion 2 follows from Lemma 3.

To prove that \( \text{int} \tilde{P}^i \neq \emptyset \) if and only if \( P^i \cap \Delta \neq \emptyset \), assume first \( \text{int} \tilde{P}^i \neq \emptyset \). From (6), \( P^i \cap \Delta \neq \emptyset \). Conversely, if \( P^i \cap \Delta \neq \emptyset \), let \( \pi \in P^i \cap \text{int} \Delta \), then \( \pi \in \text{int} \tilde{P}^i \). If \( t_i = 1 \), then \( \text{int} \tilde{P}^i = \text{int} P^i \).

To prove assertion 3, the set of no-arbitrage prices for the economy

\[
\bigcap_i S^i = \bigcap_i \text{int cone} \tilde{P}^i = \text{cone int} \tilde{P}^i.
\]

The second statement in assertion 3 follows from assertion 2 and the first statement in assertion 3.

A.6. Proof of Corollary 1

To prove assertion 1 that allows for risk neutral agents, if \( t^i < 1 \), let \( Z^i \) be constant in (9) with \( a^i < u''(z_i') < b^i \). We obtain that \( \text{int} P^i \subseteq \tilde{P}^i \). Hence \( \bigcap_i \text{int} P_i \neq \emptyset \) imply \( \bigcap_i \text{int} \tilde{P}_i \neq \emptyset \).

From Proposition 3, assertion 3, \( \bigcap_i S^i = \text{int} \mathbb{R}^k_+ \) for all \( i \) and \( \bigcap_i S^i = \text{int} \mathbb{R}^k_+ \).

A.7. Proof of Lemma 2

Let \( V \) fulfill (3), \( t = 1 \) and have no half-line. Then for every \( X \in \mathbb{R}^k \) and \( W \neq 0 \) useful, there exists \( \lambda > 0 \) such that

\[
0 < V(X + \lambda W) - V(X) \leq E_\pi (X + \lambda W - X) = \lambda E_\pi (W)
\]
for any $\pi \in P(X)$. Hence $\pi$ is a no-arbitrage price. From Proposition 3, $P(X) \subseteq \operatorname{int} P$ for any $X \in \mathbb{R}^k$. Conversely assume that $P(X) \subseteq \operatorname{int} P$ or equivalently that any $\pi \in P(X)$ is a no-arbitrage price for any $X \in \mathbb{R}^k$ and that there is a half-line. Then there exists $X \in \mathbb{R}^k$ and $W \neq 0$ useful such that $V(X + \lambda W) = V(X)$ for all $\lambda \geq 0$. Let $\pi_\lambda \in P(X + \lambda W)$. We then have

$$0 \geq E_{\pi_\lambda}(X + \lambda W - X) = \lambda E_{\pi_\lambda}(W)$$

contradicting the fact that $\pi_\lambda$ is a no-arbitrage price.

Assume that $V$ fulfills (3) and that $V$ has a half-line. Then there exists $X \in \mathbb{R}^k$ and $W \neq 0$ useful such that $V(X + \lambda W) = V(X)$ for all $\lambda \geq 0$. Let $\pi_\lambda \in P(X + \lambda W)$. We then have

$$0 \geq E_{\pi_\lambda}(u(X + \lambda W) - u(X)) \geq E_{\pi_\lambda}(u'(X + \lambda W)\lambda W).$$

Since $W$ is useful, from Proposition 2, $E_{\pi_\lambda}(u'(X + \lambda W)W) = 0$.

Assume now that $P(X) \subseteq \operatorname{int} \Delta$ for any $X \in \mathbb{R}^k$. Since $\pi_\lambda \subseteq \operatorname{int} \Delta$ and $u' > 0$, $W_+ \neq 0$ and $W_- \neq 0$. If $a < u'(x)$ or $u'(x) < b$ for all $x$, then we have $0 > aE_{\pi_\lambda}(W_+) - bE_{\pi_\lambda}(W_-)$ contradicting (8) of Remark 1.

Let us now show that if $V$ has no-half line, then $P(X) \subseteq \operatorname{int} \Delta$ for any $X \in \mathbb{R}^k$. Indeed if $V$ has no-half line, for any $X \in \mathbb{R}^k$ and any $W \in \mathbb{R}^k$ useful, there exists $\lambda > 0$ such that $0 < V(X + \lambda W) - V(X)$. Thus for any $\pi \in P(X)$, we have

$$0 < V(X + \lambda W) - V(X) \leq E_{\pi}(u'(X)\lambda W).$$

Hence $E_{\pi}(u'(X)W) > 0$ for any $W$ useful in particular for any $W \geq 0$, $W \neq 0$. Hence $\pi$ is strictly positive. If $V$ fulfills (2), then any $\pi \in P$ fulfills $\pi \in P(a)$, $a \in \mathbb{R}$, hence $P \subseteq \operatorname{int} \Delta$.

Assume that $V(X) = E_{\pi}(u(X))$. From assertion 2, $\pi \in \operatorname{int} \Delta$ and no risk neutrality is a sufficient condition for no-half-line. From assertion 3, if $V$ has no half-line, then $\pi \in \operatorname{int} \Delta$ and $E_{\pi}(u'(X)W) > 0$ for any non-zero useful vector $W$ and any $X \in \mathbb{R}^k$. If there is risk neutrality at infinity, then there exist $c, d$ such that $u'(x) = b$ for all $x \in ]-\infty, d]$ and $u'(x) = a$ for all $x \in [c, \infty[$. Thus we must have

$$aE_{\pi}(W_+) - bE_{\pi}(W_-) > 0, \quad \text{for all } W \neq 0 \text{ useful}.$$

However any $W \neq 0$ such that $aE_{\pi}(W_+) - bE_{\pi}(W_-) = 0$ is useful and violates the strict inequality. Hence we obtain a contradiction.

A.8. Existence of equilibrium theorems

A.8.1. A review of existence of equilibrium theorems

In order to prove Proposition 4, we start this section by recalling a theorem on existence of equilibrium with short-selling.
Theorem 1. Let $V^i$ fulfill (3) for each $i$. Then the following assertions are equivalent:

1. $\bigcap_i S^i \neq \emptyset$,
2. NUBA is fulfilled,
3. the set of individually rational attainable allocations $A$ is compact.
   
   Any of the previous assertions implies any of the following assertions:
4. there exists an individually rational efficient allocation for any distribution of initial endowments,
5. there exists an equilibrium for any distribution of initial endowments.

If $V^i$ has no half-line for every $i$, then assertions 1–5 are equivalent and furthermore, any equilibrium price is a no-arbitrage price.

Proof. See e.g. Page and Wooders [28], Dana et al. [7]. \hfill \Box

Theorem 1 is particularly useful when the utilities are strictly concave.

A.8.2. Strict concavity of $V$

We now provide necessary and sufficient condition for $V$ that fulfill (3) to be strictly concave.

Lemma 4. Let $V$ fulfill (3). Then $V$ is strictly concave if and only if $P(X) \subseteq \text{int} \Delta$ for any $X \in \mathbb{R}^k$ and $u$ is strictly concave. If $V$ fulfills (2), then $V$ is strictly concave if and only if $u$ is strictly concave and $P \subseteq \text{int} \Delta$.

Proof. Let us prove that if $P(X) \subseteq \text{int} \Delta$ for any $X \in \mathbb{R}^k$ and $u$ is strictly concave, then $V$ is strictly concave. Indeed, let $X, Y \in \mathbb{R}^k$, $X \neq Y$, $\lambda \in ]0, 1[$ and $\pi \in P(\lambda X + (1 - \lambda) Y)$. We then have

$$V(\lambda X + (1 - \lambda) Y) = E_\pi (u(\lambda X + (1 - \lambda) Y)) + c(\pi)$$

$$> \lambda E_\pi (u(X)) + (1 - \lambda) E_\pi (u(Y)) + c(\pi)$$

$$\geq \lambda V(X) + (1 - \lambda) V(Y),$$

proving the desired assertion. Conversely if $V$ is strictly concave, then restricting attention to constants, we first obtain that $u$ is strictly concave. As $V$ has no half-line, from the proof of Lemma 2, we obtain that $P(X) \subseteq \text{int} \Delta$ for any $X \in \mathbb{R}^k$. Clearly if $V$ fulfills (2), $P \subseteq \text{int} \Delta$. \hfill \Box

A.9. Proof of Proposition 6

The proof of assertion 1 is as that of assertion 1 of Proposition 1 in Dana and Le Van [8]. We now prove assertion 2. Assume on the contrary that there exist an efficient allocation $(X^i)_{i=1}^m$ for some distribution of endowments and a feasible trade $W^1, \ldots, W^m$ which satisfy $E_\pi(W^i) > 0$ for all $i$ and $\pi \in \tilde{P}^i$. For any $i$, for any $\pi^i \in P^i$ and $Z^i \in \mathbb{R}^k$, we have

$$\sum_s \pi^i_s u^i(z^i_s) w^i_s > 0.$$ 

In particular, we have, for any $\pi \in \arg \min_{\pi \in P^i} E_\pi (u^i(X^i + W^i)) + c^i(\pi)$,

$$V^i(X^i + W^i) - V^i(X^i) \geq c^i(\pi),$$

contradicting the Pareto optimality of $(X^i)_{i=1}^m$. 

To prove assertion 3, let \((E_i)^m\) be fixed. For any \((z^i) \in U((E_i)^m)\), there exists \((X^1, X^2, \ldots, X^m) \in A((E_i)^m)\) such that
\[
V^i(E^i) \leq z^i \leq V^i(X^i), \quad \text{for all } i.
\] (12)

From assertion 1, if there exists an efficient allocation, there exists \(\tilde{\pi} \in \bigcap_i \tilde{P}^i\). Hence there exists \((\tilde{X}^i, \pi^i)\) with \(\pi^i \in P^i\) such that \(\tilde{\pi}_j := \frac{u^j(\tilde{x}^i)\pi^i_j}{E_{\pi^i}(u^j(\tilde{x}^i))}\) for all \(i\). Let us show that if \((X^1, X^2, \ldots, X^m) \in A((E_i)^m)\), then \(E^{\tilde{\pi}}(X^i)\) is bounded. We first show that it is bounded below.

Indeed
\[
V^i(X^i) = \min_{\pi^i} E_{\pi^i} u^i(X^i) + c^i(\pi) \leq E_{\pi^i} u^i(X^i) + c^i(\pi^i)
\]
\[
\leq E_{\pi^i}(u^i(\tilde{X}^i)) + E_{\pi^i}(u^j(\tilde{X}^i))(X^i - \tilde{X}^i) + c^i(\pi^i)
\]
\[
= E_{\pi^i}(u^i(\tilde{X}^i)) + E_{\pi^i}(u^j(\tilde{X}^i) E_{\tilde{\pi}}(X^i - \tilde{X}^i) + c^i(\pi^i).
\]

Thus,
\[
m^i = \frac{V^i(E^i) - c^i(\pi^i) - E_{\pi^i}(u^i(\tilde{X}^i))}{E_{\pi^i}(u^i(\tilde{X}^i))} + E_{\tilde{\pi}}(\tilde{X}^i) \leq E_{\tilde{\pi}}(X^i).
\]

Since for all \(i\), \(E^{\tilde{\pi}}(X^i)\) is bounded below by \(m^i\), it is bounded above by \(M^i = E^{\tilde{\pi}}(E) - \sum_{l \neq i} m^l\).

From (12), we thus have
\[
z^i \leq V^i(X^i) \leq E_{\pi^i}(u^i(\tilde{X}^i)) + (M^i - E_{\tilde{\pi}}(X^i)) E_{\pi^i}(u^i(\tilde{X}^i)) + c^i(\pi^i) \quad \text{for all } i
\]

and \(U((E_i)^m)\) is bounded.

A.10. Proof of Proposition 7

The first assertion is proven as assertion 1 of Proposition 6. The equivalence between the two assertions follows from Samet [32], \(P^i(\tilde{X}^i)\) being compact for every \(i\).

References