Government spending and fiscal policy stabilizing rules*

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Abstract

Considering a finance constrained economy, where indeterminacy and sunspot fluctuations prevail due to constant structural public expenditures financed by income taxation, we discuss the stabilization role of additional cyclical labour/capital income tax rates.

We find that sufficiently procyclical labor and capital income tax rates are able to stabilize locally the economy, eliminating business cycles driven by self-fulfilling prophecies and restoring saddle path stability. However, there is at least another steady state with a lower level of output, that is either a source or indeterminate and Hopf bifurcations may occur. Hence, depending on expectations, the economy may end up converging to a a lower level of output and it is not completely insulated from instability linked to volatile expectations. Finally, we show that in the absence of a fixed level of government spending and of countercyclical tax rates the steady state is unique and saddle path stable.

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1 Introduction

Fiscal policy can influence macroeconomic (in)stability mainly through two channels. First, a considerable fraction of tax revenues are used to finance government commitments which are independent of the business cycle. Second, governments can deliberately use public spending and tax instruments to offset business cycles fluctuations. In fact, several papers have shown that countercyclical tax rates on income, used to finance a constant flow of government spending, may generate equilibrium multiplicity, indeterminacy and bifurcations, thus having a destabilizing effect on the economy by triggering endogenous expectations-driven cycles (sunspots). See, for instance, Schmitt-Grohé and Uribe (1997) Pintus (2004) and Gokan (2006). However, other types of fiscal policy rules may be able to stabilize the economy. Guo and Lansing (1998) and Guo (1999) show that progressive income or labor taxation bring saddle path stability and eliminate indeterminacy in an environment where the latter would prevail due to increasing returns to scale. However, there is not an integrated study of these two types of fiscal policies: those that introduce instability because of the need to finance a minimum of public expenditure, constant along business cycles (generating countercyclical tax rates), and those that can promote stability and potentially offset the instability created by the need to finance the referred minimum of public services. Moreover, governments use several tax instruments and tax differently labor and capital incomes. However, the comparison of the cyclical properties of labor and capital income tax rates able to bring saddle path stability, in an economy where indeterminacy would prevail in the absence of their cyclicality, is a question not yet addressed in the related literature. Our work fills these gaps.

We consider a finance constrained economy, as in Woodford (1986) and Grandmont et al. (1998) and discuss how equilibrium indeterminacy and expectations driven fluctuations emerge due to the existence of constant government spending commitments financed by (countercyclical) general income taxation. In this context, we introduce additional government spending, whose amount changes along the cycle and which is financed by additional (cyclical) income tax rates. Moreover, these cyclical tax rates may be different according to whether income comes from the use of labor or capital services. Hence, we consider cyclical specific tax rates on labour and capital

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1Note that, in models of representative agents, progressive taxation has similar effects to those of procyclical tax rates in terms of local dynamics.

2For instance, tax rates on profits or corporate taxes are a form of capital income taxation, and social insurance contributions or payroll taxes represent specific labor income taxation. There are typically different tax codes for this two types of taxes.
income and discuss which are the cyclical properties of these tax rate rules that are able to stabilize the economy with respect to endogenous business cycles.

We find that sufficiently procyclical tax rates on capital and/or labor income (and therefore procyclical government spending) bring local saddle path stability. These results confirm the insights of previous research, according to which procyclical labor tax rates promote determinacy. However, our new finding that sufficiently procyclical capital tax rates stabilize locally the economy, supporting also the traditional Keynesian view on taxation,\textsuperscript{3} rehabilitates the role of capital income taxation as a local stabilization instrument. We also find that when labor(capital) income tax rates are sufficiently procyclical, local saddle path stability can be achieved with flat or even counter-cyclical capital(labor) income tax rates. This result, which indicates that labor and capital taxation can be seen as substitutable local stabilization tools, shows that governments, whose aim is to locally stabilize the economy, do have a choice among different combinations of procyclical and counter-cyclical labor and capital income tax rates. However, we show that the need to finance constant government spending leads to steady state multiplicity which cannot be eliminated with cyclical labor and capital income tax rates. Moreover, when a sufficiently procyclical tax rate is able to make the steady state under study locally saddle stable, there exists another steady state with a lower level of output, around which sunspots may emerge, so that global stability is not achieved. These results show that, once the government, obliged to finance a constant minimum level of public services along the business cycle, induces instability, we will hardly find simple tax rules able to globally insulate the economy from expectation driven cycles. Finally, we show that an economy without fixed government spending and with no countercyclical tax rates (implying flexible procyclical government spending) has a unique steady state that is saddle path stable.

Our analysis is important for the current economic policy agenda discussing the (de)stabilizing effects of the structural public expenditures and which type of income should be taxed and how. It is also relevant to the current debate on fiscal consolidation policies. European peripheric countries, in the context of current financial and economic crisis, are struggling to balance the public budget, having imposed severe cuts in government spending and increased tax rates. Many economic analysts, fearing that this increase in tax rates would reinforce the crisis and create instability, recommended

\textsuperscript{3} According to standard Keynesian models the government should lower (increase) tax rates in bad (good) times in order to stabilize the business cycle, reducing the possible costs of fluctuations. On this issue see also Ljungqvist and Uhlig (2000).
instead a cut in tax rates on labor and capital income. Our results show that although these policies may be able to reduce local instability they do not eradicate the possibility of a deeper crisis and further instability, due to stronger pessimistic self-fulfilling expectations about future income, that will shift the economy to the lower activity steady-state. Thus, while governments are not able to avoid countercyclical tax rates and to substitute fixed by flexible procyclical public spending, managing expectations in order to avoid pessimistic beliefs seems to be an essential complement to procyclical tax rate rules.\footnote{Indeed the importance of (self-fulfilling) expectations has been clearly recognized by policy makers of the European periphery, that are determined to change markets expectations and perceptions in order to restore credibility and confidence. See Gaspar (2012).} Policy implications should however be taken carefully, since the model considered is quite stylized and more research on these issues is still needed.

The rest of the paper paper is organized as follows. In the next section we present the model considered, obtain the perfect foresight equilibria and define the steady state. In section 3 we study the local stability properties of the model and discuss local dynamics in the absence of cyclicality of tax rates on labor and capital income. In section 4, ensuring the existence of a normalized steady state, we discuss the role of specific tax rates on capital and labor income as local stabilization instruments. Steady state multiplicity and global stability are adressed in section 5 and we discuss our results in section 6. Finally, in section 7 we provide some concluding remarks.

## 2 The Model

The model here considered extends the Woodford (1986)/Grandmont et al. (1998) framework introducing public spending financed by taxation and externalities in preferences. We consider a perfectly competitive monetary economy with discrete time $t = 1, 2, \ldots, \infty$ and heterogeneous infinitely lived agents of two types: workers and capitalists. Both consume the final good, but only workers supply labor. There is a financial market imperfection that prevents workers from borrowing against their wage income and workers are more impatient than capitalists, i.e. they discount the future more than the latter. So, in a neighborhood of a monetary steady state, capitalists hold the whole capital stock and no money, whereas workers save their wage earnings through money balances and spend it in consumption in the following period. The final good, which can be used for consumption or capital investment, is produced by firms under a Cobb-Douglas technology characterized by constant returns to scale. We introduce two types of public spending in this
framework: constant structural government spending that reflects commitments independent of the business cycle, and cyclical government spending whose value may vary along business cycles. The latter is considered "wasteful" public spending and is financed by specific variable labor and/or capital income taxes. In contrast, structural spending is financed by general income taxation and has utility, i.e., we introduce structural government spending positive externalities on preferences. The detailed description of the model and the perfect foresight equilibrium are provided below. In order to focus our analysis on instability linked to autonomous volatility of expectations, we disregard uncertainty about the economic fundamentals, considering stationary preferences, technology and fiscal policy rules.

2.1 Production

In each period \( t = 1, 2, \ldots, \infty \), both capital \( k_{t-1} > 0 \) and labor \( l_t > 0 \) are used to produce output \( y_t \) under a Cobb-Douglas technology with constant returns to scale,

\[
y_t = k_t^s l_t^{1-s}
\]

where \( s \in (0,1) \) represents the capital share of income. From profit maximization, the marginal productivities of capital and labor are respectively equal to the real rental rate of capital (i.e. the real interest rate) \( \rho_t \) and the real wage \( \omega_t \), i.e.

\[
\rho_t = sk_t^{s-1}l_t^{1-s}, \quad \omega_t = (1-s)k_t^s l_t^{-s}.
\]

Therefore there are no economic profits at equilibrium, i.e., \( y_t = \omega_t l_t + \rho_t k_{t-1} \).

2.2 The Government

As usually assumed in the literature the government runs a balanced budget. The main novelty is that we consider two types of public expenditure (and revenue). On one hand, the government collects every period, through general income taxation, a constant amount of fiscal revenue, used to finance an equivalent flow of real government expenditures, \( \overline{G} \geq 0 \). Therefore, letting \( \tau y_t \in (0,1) \) denote the income tax rate, we have:

\[
\tau y(y_t) = \tau y_t = \frac{\overline{G}}{y_t}
\]
Note that the tax rate $\tau_y$ is countercyclical, decreasing (increasing) when output increases (decreases) with an elasticity of $-1$. The level of spending, $G$, reflects the views of government and society on the appropriate size of unavoidable government expenditures, that should remain constant along business cycles. We further assume that this amount of spending corresponds to public services that positively influence households’ utility.

On the other hand, we also introduce possibly wasteful variable government expenditures that are financed by additional specific labor and/or capital income taxes, whose aim is to stabilize the economy. Accordingly, these taxes vary with the level of income/output in the economy, being taken as given by individuals. Moreover we consider that different tax rules may apply to income generated by labor or capital services. We thus define specific tax rates on labor income $\tau_{Lt} \in [0, 1)$ and on capital income $\tau_{Kt} \in [0, 1)$, at period $t$, that are given respectively by

$$
\tau_L (y_t) = \tau_{Lt} \equiv \mu_L y_t^\phi_L, \quad \text{and} \quad \tau_K (y_t) = \tau_{Kt} \equiv \mu_K y_t^\phi_K
$$

The parameter $\phi_i = \frac{d\tau_i}{dy} \in R$ for $i = L, K$, represents the elasticity of the tax rate with respect to total income or output. When $\phi_i < 0$ the tax rate decreases when the level of output expands, i.e., the tax rate moves countercyclically. The case of $\phi_i > 0$ corresponds to the case where the tax rate increases with output, i.e. the tax rate is procyclical. For $\phi_i = 0$ the tax rate is constant, so that cyclicality of tax rates is absent.$^5$ The parameters $\mu_L \in (0, 1)$ and $\mu_K \in (0, 1)$ represent the tax rates when $y = 1$. Note that, in any period $t$, the total tax rate on labor income is $\tau_y + \tau_{Lt}$ and the total (real) tax revenues from labor income are $(\tau_y + \tau_{Lt}) \omega_l l_t$, while the total tax rate on capital income and total real tax revenues from capital income are given, respectively, by $\tau_y + \tau_{Kt}$ and $(\tau_y + \tau_{Kt}) \rho_t k_{t-1}$.

Summing up, there are general taxes on income (from now on referred simply as income taxes) used to finance a constant flow of government expenditures $G$ and, on top of that, the government can use different specific tax rates on capital and labor (from now on referred simply as tax rates on capital or labor income), keeping the budget balanced. Accordingly, we have:

$$
G_t = \tau_L (y_t) \omega_l l_t + \tau_K (y_t) \rho_t k_{t-1} + \mathcal{G}
$$

Note that we distinguish between cyclical public expenditures ($G_t - \mathcal{G}$), whose amount may vary along the business cycle, and structural government expenditures. $^5$This specification nests most cases considered in the literature. For example, the case considered in Gokan (2006), Pintus (2004) and Schmitt-Grohé and Uribe (1997) where a constant amount of public expenditures is financed by taxes corresponds to the case where $\phi_i = -1$ (as in (4)).
spending $\overline{G}$, that does not respond to economic fluctuations.\(^6\) This seems to be better suited to deal with current concerns of countries undergoing problems of sovereign debt, where the appropriate size of government has also been under discussion.

For future reference we introduce the following notation:

$$a_L(y) \equiv \phi_L \frac{\tau_L(y)}{1 - \tau_L(y)} \in (-\infty, +\infty) \quad (7)$$

$$a_K(y) \equiv \phi_K \frac{\tau_K(y)}{1 - \tau_K(y)} \in (-\infty, +\infty) \quad (8)$$

$$b_L(y) \equiv \frac{1 - \tau_L(y)}{1 - \tau_L(y) - \tau_y(y)} \in [1, +\infty) \quad (9)$$

$$b_K(y) \equiv \frac{1 - \tau_K(y)}{1 - \tau_K(y) - \tau_y(y)} \in [1, +\infty) \quad (10)$$

### 2.3 Workers

We introduce government spending externalities,\(^7\) assuming that structural government expenditures are seen as useful public consumption that generates utility, so that total consumption utility depends also on $\overline{G}$.

The behavior of the representative worker can be summarized by the maximization of the utility function given in (11), subject to the constraints in (12) below:

$$U(c_{t+1}^{w}, \overline{G}, l_t) \equiv c_{t+1}^{w} \frac{g(\overline{G})}{B} - l_t^{\gamma} / \varepsilon_\gamma$$

where $l$ are hours worked with $l \in \{0, \tilde{l}\}$, $\tilde{l}$ is the worker’s time endowment exogenously specified and possibly infinite, $\varepsilon_\gamma > 1$ with $1 / (\varepsilon_\gamma - 1) > 0$ representing the individual elasticity of labor supply, $B > 0$ is a scaling parameter, and $g(\overline{G}) > 0$, with $g(0) = 1$, represents (structural) public consumption externalities.\(^8\)

\(^6\)Note that from (4) and (6), both structural and cyclical budgets are balanced. However, if we had assumed that the government only balances the total budget our results would not change significantly.


\(^8\)Note that, if $g'(\overline{G}) > 0$, $c_t^{w}$ and $\overline{G}$ are Edgeworth complements, i.e. the marginal utility of individual consumption is increasing in $\overline{G}$ ($\overline{G}^2 U(\overline{G}) > 0$). Ni (1995) provides empirical support for Edgeworth complementarity between private and public consumption. However our results do not depend on that.
The worker’s constraints are given as follows:

\[ p_{t+1} c_{t+1}^w = m_t = (1 - \tau_L(y_t) - \tau_y(y_t)) w_t l_t \]  

(12)

with \(1 - \tau_L(y_t) - \tau_y(y_t) > 0\), \(w_t\) is the nominal wage at period \(t\), \(m_t\) represents money holdings at the beginning of period \(t + 1\) and \(p_{t+1}\) represents the expectation for the price of the final good which, under perfect foresight, ends up being its market equilibrium level at \(t + 1\). Workers, when solving their maximisation problem take tax rates as given.

The solution for \(c_{t+1}^w\) and \(l_t\) of this problem is given by

\[ p_{t+1} c_{t+1}^w = (1 - \tau_L(y_t) - \tau_y(y_t)) w_t l_t, \]

(13)

with the intertemporal trade-off between future consumption and leisure:

\[ c_{t+1}^w g(G) / B = l_t^\gamma, \]

while the demand for money holdings satisfies \(m_t = (1 - \tau_L(y_t) - \tau_y(y_t)) w_t l_t\).

From (13), we can see that labor is a non predetermined variable, whose current value depends on future consumption for \(t + 1\), planned by the worker at time \(t\), which is influenced by expectations for \(p_{t+1}\). Therefore, there is a priori room for fluctuations in employment and output driven by changes in expectations.

### 2.4 Capitalists

The representative capitalist maximizes the log-linear lifetime utility function

\[ \sum_{t=1}^{\infty} \beta^t \ln c_t^c \]

subject to the budget constraint \(c_t^c + k_t = (1 - \delta + (r_t/p_t)(1 - \tau_K(y_t) - \tau_y(y_t)))[k_{t-1}]\), with \(1 - \tau_K(y_t) - \tau_y(y_t) > 0\) and where \(c_t^c\) represents his consumption at period \(t\), \(k_t\) is the capital stock held at the end of period \(t\) by capitalists, \(\beta \in (0, 1)\) his subjective discount factor, \(r_t\) the nominal interest rate and \(\delta \in (0, 1)\) the depreciation rate of capital.\(^9\) Capitalists also take tax rates as given. Solving the capitalist’s problem we obtain the capital accumulation equation:

\[ k_t = \beta [1 - \delta + (r_t/p_t)(1 - \tau_K(y_t) - \tau_y(y_t))] k_{t-1}. \]  

(14)

### 2.5 Equilibrium

Equilibrium on labor and capital services markets requires \(\omega_t = w_t/p_t\), \(\rho_t = r_t/p_t\). Considering that \(m > 0\) is the constant money supply, from (12) we

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\(^9\)Since in our framework tax rates depend on aggregate variables (see (4), and (5)) individuals, being atomistic, take tax rates as given.

\(^10\)We do not introduce government spending externalities into capitalists’ preferences because, since they have a log-linear utility function, such externalities would not affect the dynamics nor the steady state.
have that at the monetary equilibrium \((1 - \tau_L(y_t) - \tau_y(y_t))w_t l_t = m\) in every period \(t\) and, therefore, \(c_t^{w+} = (1 - \tau_L(y_{t+1}) - \tau_y(y_{t+1}))\omega_{t+1} l_{t+1}\). Hence, using (13)-(14), (1)-(3) and (4)-(5) we have the following definition:

**Definition 1** A perfect foresight intertemporal equilibrium is a sequence \((k_{t-1}, l_t) \in \mathbb{R}^2_{++}, t = 1, 2, ..., \infty\) that, for a given \(k_0 > 0\), satisfies

\[
(1 - \tau_L(y_{t+1}) - \tau_y(y_{t+1}))\omega_{t+1} l_{t+1} g(G)/B = l_t^\gamma
\]

\[
k_t = \beta [1 - \delta + \rho_t(1 - \tau_K(y_t) - \tau_y(y_t))] k_{t-1}
\]

with \(y, \omega, \rho\) given respectively by (1)-(3) and \(\tau_y(y), \tau_L(y)\) and \(\tau_K(y)\) are given respectively by (4) and (5), satisfying \(z_K(y) \equiv 1 - \tau_K(y) - \tau_y(y) > 0\) and \(z_L(y) \equiv 1 - \tau_L(y) - \tau_y(y) > 0\) and \(g(0) = 1\).

Equations (15) and (16) represent, respectively, the intertemporal trade-off between consumption and leisure and capital accumulation. They determine the dynamics of this economy through a two-dimensional dynamic system with only one predetermined variable, the capital stock \(k\). Indeed, the amount of capital used in production at period \(t\), \(k_{t-1}\), is a variable determined by past actions. Employment \(l_t\), on the contrary, is affected by expectations of future events as explained before.

### 2.6 Steady State

A steady state \((k, l)\) is a stationary solution \(k_t = k_{t-1} = k\) and \(l_{t+1} = l_t = l\) of (15)-(16), satisfying Definition 1. Using (1)-(3), note that \(\omega l = (1 - s) y\) and \(\rho = s (y/l)^{(s-1)/s}\). Hence, we can write the following:

**Definition 2** A steady state of the dynamic system (15)-(16) is a pair \((k_*, l_*) \in \mathbb{R}^2_{++}\) with the corresponding level of output \(y_* = k_*^{1-s} l_*^s \in \mathbb{R}_{++}\), such that

\[
H(y_*) = \mathbb{P} \tag{17}
\]

\[
l_* = [(1 - s) y_* (1 - \tau_L(y_*) - \tau_y(y_*))]^{1/\gamma} (g(G)/B)^{1/\gamma} \tag{18}
\]

\[
k_* = (y_*/l_*^{1-s})^{1/s} \tag{19}
\]

where

\[
H(y) = y^{1-s} (z_K(y))^{\frac{1}{\gamma}} [z_L(y)]^{\frac{1}{\gamma}} \tag{20}
\]

\[
\mathbb{P} = \frac{\theta}{\beta s} \left[ \frac{g(G)}{B} \right]^{\frac{1}{\gamma}} > 0
\]

with \(\tau_y(y), \tau_K(y)\) and \(\tau_L(y)\) given in (4)-(5), \(z_K(y) \equiv 1 - \tau_K(y) - \tau_y(y) > 0\), \(z_L(y) \equiv 1 - \tau_L(y) - \tau_y(y) > 0\), and where \(g(0) = 1\) and \(\theta \equiv 1 - \beta (1 - \delta) \in (0, 1)\).
In Section 4, we ensure the existence of a steady state following the usual procedure of fixing the scale parameter $B$ at the appropriate level. The steady state may however not be unique, as discussed in Section 6.

For future reference we compute $H'(y)$, the first derivative of $H(y)$:

$$H'(y) \frac{y}{H(y)} = \frac{1-s}{s} \left[ \frac{1}{\varepsilon_\gamma} - 1 \right] - \frac{1-s}{s} \frac{\tau_L(y)\phi_L}{z_L(y)} - \frac{\tau_K(y)\phi_K}{z_K(y)} +$$

$$+ \tau_y(y) \left( \frac{1}{\varepsilon_\gamma} \frac{1}{z_L(y)} + \frac{1}{z_K(y)} \right). \quad (21)$$

Note that from (9)-(10) we have $\tau_y(y) = b_i(y) - 1$ so that, using (7)-(8) and rearranging terms, we can rewrite (21) as:

$$H'(y) \frac{y}{H(y)} = \frac{(1-s)b_L - (1-sb_K)\varepsilon_\gamma - (1-s)b_La_L - sb_Ka_K\varepsilon_\gamma}{sb_K(\varepsilon_\gamma)} \quad (22)$$

3 Local Stability properties

We now characterize the local stability properties of our dynamic system around a steady state $y^*$. We first log-linearize the system (15)-(16) around a steady state $y^*$, obtaining:

$$\begin{bmatrix} \hat{k}_t \\ \hat{t}_{t+1} \end{bmatrix} = [J] \begin{bmatrix} \hat{k}_{t-1} \\ \hat{t}_t \end{bmatrix} \quad (23)$$

where hat-variables denote percentage deviation rates from their steady-state values and $J$ is the Jacobian matrix of the system (15) and (16) evaluated at the steady state. Its trace, $T$, and determinant, $D$, are given by:

$$T = 1 + \varepsilon_\gamma - \theta (1-s) (1-a_L) b_L \quad (24)$$

$$D = \frac{1 - \theta + \theta s (1-a_K) b_K}{(1-s) (1-a_L) b_L} \varepsilon_\gamma \quad (25)$$

where, for $i = L, K$, we have $a_i \equiv a_i(y^*) = \phi_i \frac{\tau_i}{1-\tau_i} > 0$ and $b_i \equiv b_i(y^*) = \frac{1-\tau_i}{1-\tau_i - \tau_Y} > 1$, see (7)-(10), with $\tau_y$, $\tau_L$ and $\tau_K$ denoting, respectively, the tax rate on income, and the tax rates on labor and capital income, all evaluated at the steady state under analysis $y^*$, i.e., $\tau_y \equiv \frac{\tau_Y}{y^*}$, $\tau_L \equiv \mu_L y^{\phi_L}$ and $\tau_K \equiv \mu_K y^{\phi_K}$. See (4)-(5).
The local stability properties of the model are determined by the eigenvalues of the Jacobian matrix $J$ or, equivalently, by its trace, $T$, and determinant, $D$, which correspond respectively to the product and sum of the two roots (eigenvalues) of the associated characteristic polynomial $Q(\lambda) \equiv \lambda^2 - \lambda T + D$.

In what follows, as typically done in Woodford economies, we assume that $0 < \theta (1-s) < s < 1/2$, i.e., that the period is short so that $\theta$ is small, and that $s$ is also small. Moreover, in accordance with empirical studies and following Lloyd-Braga, Modesto and Seegmuller (2014), we assume that after-tax gross real capital income, $[1 - \delta + \rho_t (1 - \tau_k(y_t) - \tau_y(y_t))] k_{t-1}$, is increasing with capital and that the after-tax real wage bill, $(1 - \tau_L(y_{t+1}) - \tau_y(y_{t+1})) \omega_{t+1} l_{t+1}$, is increasing in labor. These two assumptions imply respectively that, at the steady state under analysis, $1 - \theta s (1 - b_K) > \theta b_K a_K$ and $(1 - a_L) > 0$.

All these assumptions are summarized below in Assumption 1 and we consider them satisfied in the rest of the paper.

**Assumption 1**

1. $0 < s < 1/2$ and $0 < \theta < s/(1-s)$
2. $a_K < \bar{a}_K$, with $\bar{a}_K \equiv \frac{1+\theta s(b_K-1)}{\theta b_K} > 0$
3. $a_L < 1$

### 3.1 Analytical Results

Analytical results are easier to obtain with the support of Figure 1, where we have represented in the plane $(T, D)$ three lines relevant for our purpose: the line $AC$ ($D = T - 1$) where a local eigenvalue is equal to 1; the line $AB$ ($D = -T - 1$), where one eigenvalue is equal to -1; and the segment $BC$ ($D = 1$ and $|T| < 2$) where two eigenvalues are complex conjugates of modulus 1. When $T$ and $D$ fall in in the interior of triangle ABC the steady state is a sink (both eigenvalues with modulus lower than one), i.e., asymptotically stable. In this case, given the present context where only capital is a predetermined variable, the steady state is locally indeterminate\footnote{Indeterminacy occurs when the number of eigenvalues strictly lower than one in absolute value is larger than the number of predetermined variables.} and, as known, there are infinitely many stochastic endogenous fluctuations (sunspots) arbitrarily close to the steady state. In all other cases the steady state is locally determinate. It exhibits saddle path stability (one eigenvalue with modulus higher than one and one eigenvalue with modulus lower than
one) when $|T| > |D + 1|$ and it is an unstable source (both eigenvalues with modulus higher than one) in the remaining regions.

Straightforward computations show that, under Assumption 1, we always have $D > 0$ and $D > -T - 1$. Therefore, only the 3 shaded regions depicted in Figure 1 are possible. We will have a source when $D > \max\{1, T - 1\}$, a saddle when $D < T - 1$ and a sink when $T - 1 < D < 1$. Note that if, by continuously changing a parameter of the model, the values of $T$ and $D$ cross the segment $BC$, a local Hopf bifurcation generically occurs (a pair of complex conjugate eigenvalues cross the unit circle). In this case there are deterministic cycles describing orbits that lie over an invariant closed curve, surrounding the steady state, in the state space. If the Hopf bifurcation is subcritical this curve emerges when the steady state is a sink and sunspot fluctuations arbitrarily close to the steady state emerge. When the Hopf bifurcation is supercritical the invariant closed curve appears when the steady state is a source and, although sunspot equilibria that stay arbitrarily close to the steady state do not exist, there are nevertheless infinitely many equilibria exhibiting bounded stochastic fluctuations around the invariant closed curve. See, for instance, Grandmont et al (1998). Also, if, by continuously changing a parameter of the model, the values of $T$ and $D$ cross the line $AC$, a local
transcritical bifurcation generically occurs (one eigenvalue crossing the value 1). In this case, if \((T, D)\) is close enough to line \(AC\), two close steady states co-exist. These two steady states exchange stability properties as \((T, D)\) crosses line \(AC\). When \((T, D)\) is on line \(AC\) the two steady states collapse into only one. Finally note that as, under Assumption 1 we have \(D > -T - 1\), local flip bifurcations, through which an eigenvalue crosses the value -1 (with values of \(T\) and \(D\) crossing the line \(AB\)), cannot occur.

From Figure 1 and as stated above the steady state is a saddle when \(D < T - 1\). Using (24)-(25) we obtain the following Proposition:

**Proposition 1** Under Assumption 1 a steady state \(y_\ast\) is a saddle if and only if

\[
(1 - s)b_La_L + sb_Ka_K\varepsilon_\gamma > (1 - s)b_L - (1 - sb_K)\varepsilon_\gamma \equiv \Psi, \tag{26}
\]

where \(a_L, a_K, b_L\) and \(b_K\) are given by (7)-(10) and evaluated at the steady state \(y_\ast\) under analysis.

Using (22) and Proposition 1, we can immediately see that a steady state solution \(y_\ast\) is a saddle if and only if \(H'(y)\frac{\Psi}{H(y)} < 0\) at \(y_\ast > 0\). Since \(H(y_\ast) > 0\) by Definition 2, we have the following Proposition:

**Proposition 2** Under Assumption 1, a steady state solution \(y_\ast\) is:

- a saddle if and only if \(H(y)\) is decreasing at \(y_\ast\), i.e., if \(H'(y_\ast) < 0\);
- a source or a sink when \(H(y)\) is increasing at \(y_\ast\), i.e., if \(H'(y_\ast) > 0\).

### 3.2 Local dynamics in the absence of cyclical tax rates of labor and capital income

In this section we discuss local stability properties in the absence of cyclicality in labor and capital income tax rates, i.e., when \(\phi_L = \phi_K = 0\) so that, from (7)-(8), \(a_L(y) = a_K(y) = 0\). We start by considering the case of an economy without government.

\(^{12}\)We disregard the case of a saddle node bifurcation since in Section 4 we apply our local dynamics analysis to a normalized steady state whose persistence is ensured. Pitchfork bifurcations do not occur either since, as shown in Section 5, either the normalized steady state is unique or we have an even number of steady states.
3.2.1 An economy without government

Here we consider that there are no government expenditures and no taxes, so that, from (9)-(10), we have \( b_L(y) = b_K(y) = 1 \). Then, using Proposition 1, we can see that since \( \varepsilon > 1 \) the steady state is a saddle. Note also that existence and uniqueness of the steady state is always ensured. Indeed, using Definition 2 with \( g(G) = 1 \) and \( z_K(y) = z_K(y) = 1 \), we obtain \( H(y) = y^{\frac{1-s}{s}}(\frac{1}{\varepsilon^\gamma}-1) \) which is a positive function of \( y > 0 \), continuous and decreasing (since \( \varepsilon > 1 \)), with \( \lim_{y \to 0} H(y) = +\infty \) and \( \lim_{y \to \infty} H(y) = 0 \). Hence \( H(y) \) must cross the value \( \overline{H} \) for some \( y > 0 \) only once.

**Proposition 3** In the absence of government, a steady state exists, it is unique and saddle stable.\(^{13}\)

3.2.2 An economy with government

We consider now the existence of a government that does not use cyclical tax rates on labor and capital income. In this case the following Proposition applies.

**Proposition 4** In the absence of cyclicity of tax rates on capital and labor income \((\phi_K = \phi_L = 0)\), and under Assumption 1, local indeterminacy of a steady state \( y^* > 0 \) requires positive structural government expenditures, \( \bar{G} > 0 \), and, for given values of \( \mu_L \) and \( \mu_K \), a sufficiently high steady state output share of structural government spending, \( \tau_y(y^*) \).

**Proof** See the Appendix.\(^{\bullet}\)

Now, in Proposition 5 below, we establish the necessary and sufficient condition for the emergence of indeterminacy in the absence of cyclicity of tax rates.

**Proposition 5** Under Assumption 1 and Proposition 4, a steady state \( y^* \) is locally indeterminate in the absence of cyclicity in capital and labor income tax rates \((\phi_K = \phi_L = 0)\) if and only if \( \bar{G} > 0 \) and \((1-s)b_L > (1-\theta+\theta sb_K)\varepsilon_\gamma\), which implies also that \( \Psi \equiv (1-s)b_L - (1-sb_K)\varepsilon_\gamma > 0 \), where \( b_L \) and \( b_K \) are given by (9)-(10) and evaluated at the steady state \( y^* \) under analysis.

**Proof** See the Appendix.\(^{\bullet}\)

\(^{13}\)Note that in the absence of government this result still applies even if \( \theta \in (0,1) \) and \( s \in (0,1) \) do not satisfy Assumption 1. In fact we obtain that \( D > -T - 1, D > 0 \) and \( D > T - 1 \) for all \( \varepsilon_\gamma > 1 \).
Proposition 4 states that structural government spending might induce local indeterminacy in the absence of cyclical stabilization policy, while Proposition 5 states precise conditions under which this happens. As explained above, bounded endogenous fluctuations (caused by autonomous changes in expectations even when fundamentals are stationary) may emerge around the steady state considered when the later is a sink (indeterminate) or even a source (supercritical Hopf). Also, when the steady state is locally saddle stable there are no endogenous fluctuations arbitrarily close to it. However, this does not guarantee that larger endogenous fluctuations do not exist. In Section 4, departing from a situation where local indeterminacy (and thereby local cycles) would exist in the absence of cyclicality in capital and labor income tax rates, we show that by introducing sufficient procyclical tax rates the government is able to ensure local saddle path stability. However, in Section 5 we also show that under procyclical specific tax rates the steady state is not unique and another steady state where output is lower and which is a source or a sink, also exists. This means that with sufficiently procyclical tax rates, local instability associated with small volatility of expectations is eliminated, but the risk of larger fluctuations, linked to larger autonomous changes in expectations, leading to lower values of output, still exists. We conclude that tax rate procyclicality is not able to guarantee global stabilization with respect to endogenous cycles and autonomous self fulfilling volatile expectations.

4 Cyclical tax rates and local stabilization policy

In order to discuss local stabilization policy around a steady state, let us first ensure its existence, namely ensure the existence of the normalized steady state \( y_* = y_{*N} = 1 \), by fixing the scale parameter \( B \) at the appropriate level.\(^\text{14}\) Using (4)-(5) and (7)-(10), note that at the normalized steady state the following applies:

\[
\begin{align*}
\tau_L &= \mu_L, \tau_K = \mu_K, \text{and } \tau_y = \frac{G}{C} \\
a_L &= \frac{\phi_L \mu_L}{1 - \mu_L}, \quad a_K = \frac{\phi_K \mu_K}{1 - \mu_K} \\
b_L &= \frac{1 - \mu_L}{1 - \mu_L - G}, \quad b_K = \frac{1 - \mu_K}{1 - \mu_K - G}
\end{align*}
\]

\(^\text{14}\)This is the usual procedure. See, for instance, Cazzavillan et al. (1998).
From Definition 2, it is immediate to obtain Proposition 6 below, which ensures existence of the normalized steady state.

**Proposition 6 Normalized Steady State:** Define

\[
\begin{align*}
  l_{*N} & \equiv \left[ (1 - s) (1 - \mu_L - \overline{G}) \right]^{\frac{1}{\gamma}} \left( g(\overline{G}) / B \right)^{\frac{1}{\gamma}} = \\
  k_{*N} & \equiv \frac{l_{*N}}{s} \text{ and} \\
  y_{*N} & \equiv 1
\end{align*}
\]

Then \((k_*, l_*) = (k_{*N}, l_{*N})\) with the corresponding level of output \(y_* = y_{*N}\) is a (normalized) steady state of the dynamic system (15)-(16) if and only if

\[
B = B_* \equiv \left[ 1 - \mu_K - \overline{G} \right]^{\frac{1}{1 - \gamma}} (1 - \mu_L - \overline{G}) (1 - s) g(\overline{G}) \left[ \frac{\beta s}{\theta} \right]^{\frac{1}{1 - \gamma}}.
\]

Moreover, at the normalized steady state, the tax rates \(\tau_L, \tau_K, \tau_y\) and \(a_L, a_K, b_L\) and \(b_K\) are given by (27) with \(1 - \mu_L - \overline{G} > 0\) and \(1 - \mu_K - \overline{G} > 0\).

Assuming that conditions of Proposition 6 are satisfied, we discuss now the role of cyclical tax rates on local stabilization, departing from a situation where, in the absence of cyclical tax rates, the normalized steady state is locally indeterminate due to the existence of structural government spending \(\overline{G} > 0\), as stated in Proposition 4 and Proposition 5. Local stabilization policy will be discussed considering different possible values for \(\phi_L\) and \(\phi_K\) (hence, different values of \(a_L\) and \(a_K\))^15 for fixed and empirically plausible values of the other parameters, consistent with Assumption 1, Proposition 5, Proposition 6 and (27). We consider \(\beta = 0.99\) and \(\delta = 0.025\), in line with most calibrations used in the business cycle literature for quarterly data. Hence, \(\theta = 0.03475\), and we consider \(s = 0.35\), so that \(\theta (1 - s) < s < 0.5\), as required by Assumption 1. Concerning tax parameters we set \(\tau_y (= \overline{G}) = 0.28\), \(\mu_L = 0.20\) and \(\mu_K = 0.06\), implying a total labor income tax rate of 0.48 and a total capital income tax rate of 0.34. These two last figures are in line with the ones reported in Mendonza et al. (1994) for European countries and are also consistent with reported ratios of tax revenues in GDP around 40% for the euro area in 2011.\(^{16}\) To fix the value of \(\overline{G}\) we considered that only 55% of government spending was non wasteful. Using again values for

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\(^{15}\)As \(\phi_K\) and \(\phi_L\) do not influence \(l_{*N}\), \(k_{*N}\) and \(y_{*N}\), existence of the normalized steady state is persistent and always ensured.

\(^{16}\)See Eurostat: Statistic in focus 55/2012. Indeed we have that 0.48(1 - s) + 0.34s = 0.43.
Europe, where on average government spending represents 51% of GDP,\textsuperscript{17} we arrived at a value of 0.28.\textsuperscript{18} Finally we consider $\varepsilon_\gamma = 1.01$.$^{19}$

4.1 Cyclical labor income tax rates

We start by characterizing the local stability properties of a steady state in terms of $a_L$ when cyclicality of the capital income tax rate is absent, i.e. $\phi_K = 0$ so that $a_K = 0$. Using Proposition 1 with $a_K = 0$, we see that the steady state is a saddle with $D > T - 1$ when $a_L > \frac{(1-s)b_L - (1-sb_K)\varepsilon_\gamma}{(1-s)b_L} \equiv a_L^T$. Hence, it will be a source when $a_L < a_L^T$ and $D > 1$. From (25) we obtain that $D < 1$ when $a_L > a_L^H \equiv \frac{(1-s)b_L - (1-\theta + \theta sb_K)\varepsilon_\gamma}{(1-s)b_L}$. Hence the steady state is a source when $a_L^H < a_L < a_L^T$. In all other cases, i.e. when $a_L < a_L^H$ it will be a sink. Accordingly, the following applies:

**Proposition 7** Under Proposition 6, consider the normalized a steady state $\gamma_N$ and evaluate $b_L$, $b_K$ and $a_L$ at this steady state as in (27). Under Assumption 1, let $b_K < 1/s$ and define $a_L^T$ and $a_L^H$ as:

\[ a_L^T \equiv \frac{\Psi}{(1-s)b_L} \]  \hspace{1cm} (28) \]
\[ a_L^H \equiv 1 - \frac{(1-\theta + \theta sb_K)\varepsilon_\gamma}{(1-s)b_L} \]  \hspace{1cm} (29) \]

where $\Psi \equiv (1-s)b_L - (1-sb_K)\varepsilon_\gamma$, and $a_L^T > a_L^H$.\textsuperscript{20} Then, for $\phi_K = 0$ (so that $a_K = 0$), we have that the normalized steady state is

- a source (unstable) if and only if $a_L^H < a_L < a_L^T$;
- a saddle if and only if $a_L > a_L^T$;

\textsuperscript{17}See Eurostat: Statistic in focus 33/2012.
\textsuperscript{18}This figure includes health (7.4\%) and education (5\%) expenditures, defense, public order and safety, environmental protection, housing and culture (5.8\%), general public services (6.8\%) and 70\% of economic affairs (3\%). It excludes social protection expenditures (20\%) since in this model we not address redistribution policies.
\textsuperscript{19}We chose a value consistent with the indivisible labor formulation of Hansen (1985) and Rogerson (1988) where $\varepsilon_\gamma = 1$. Note that there is no definite consensus on the correct value of this elasticity. For a discussion on the possible values of this parameter see for instance Ljungqvist and Sargent (2011), where it is claimed that "strong differences of opinion about the labor supply elasticity prevail".
\textsuperscript{20}See the proof of Proposition 5, where we have shown that under Assumption 1 $(1-\theta + \theta sb_K)\varepsilon_\gamma > (1-sb_K)\varepsilon_\gamma$. Hence, $a_L^T > a_L^H$. The condition $b_K < 1/s$ ensures that $a_L^T < 1$, so that the set of parameter values ensuring local saddle stability under Assumption 1 is not empty.

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• a sink (indeterminate) if and only if $a_L < a_L^H$.

Note that when $a_L$ crosses the critical value $a_L^T$ (i.e., when $\phi_L$ crosses the value $\phi_L^T \equiv a_L^T(1-\mu_L)/\mu_L$, see (27),) a transcritical bifurcation generically occurs. Also when $a_L$ crosses the critical value $a_L^H$ (when $\phi_L$ crosses the value $\phi_L^H \equiv a_L^H(1-\mu_L)/\mu_L$) a Hopf bifurcation generically occurs.

Proposition 7 states that if at the normalized steady state the condition $a_L > a_T^L$ is satisfied then the normalized steady state is a saddle. Under Proposition 5, stating conditions for structural government spending to induce local indeterminacy in the absence of cyclical stabilization tax rules, we have that $\Psi > 0$ so that $a_T^L > 0$. Hence, if the government is willing to use cyclical labor income tax rates in order to (locally) stabilize the economy and ensure local saddle path stability, this tax rate should be sufficiently procyclical, i.e., using (27), we must have $\phi_L > \phi_L^T \equiv a_L^T(1-\mu_L)/\mu_L$. Accordingly, we have the following result:

**Proposition 8** Let Assumption 1 and the conditions of Proposition 6 and Proposition 5 be verified, and further assume that $b_K < 1/s$ at the normalized steady state. Then, in the absence of cyclicality in capital income tax rates, $\phi_K = a_K = 0$, a sufficiently procyclical labor income tax rate, $\phi_L > \phi_L^T \equiv a_L^T(1-\mu_L)/\mu_L$ with $a_L^T$ given in (28), is able to guarantee local saddle path stability of the normalized steady state and eliminates local indeterminacy caused by positive structural government spending.\(^{21}\)

This Proposition tells us that a sufficiently procyclical labor income tax rate eliminates local endogenous fluctuations that would exist otherwise due to structural government spending. Note however that when $\phi_L > \phi_L^T$ the total tax rate faced by workers, $\tau_L(y) + \tau_y(y)$, will only be procyclical, i.e., increasing in $y$, if $\phi_L \tau_L(y) > \tau_y(y)$. For instance, under our calibration, $a_L^T = 0.494$ so that, since $\mu_L = 0.20$, using (27) we obtain $\phi_L^T = 1.974$. Hence, with $\phi_L > 1.974$ the government guarantees local saddle path stability and, given $\bar{G} = 0.28$ as under our numerical example, we have that $\phi_L \tau_L(y) > \tau_y(y)$ at the normalized steady state, i.e. $\phi_L \mu_L > \bar{G}$ (see (27)). Therefore, in this calibrated example, the total tax rate faced by workers is procyclical around the normalized steady state when the government chooses a sufficiently procyclical tax rate on labor income in order to eliminate local instability created by structural government spending.

To understand why a sufficiently procyclical labor income tax rate eliminates local indeterminacy and cycles driven by self-fulfilling volatile expec-

\(^{21}\)Of course, since under Assumption 1 $a_L < 1$, $\phi_L$ cannot be too high, i.e., $\phi_L < 1-\mu_L/\mu_L$. 

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tations, consider that at some period $t$, departing from a steady state equilibrium, agents expect an increase in future output. This leads to a decrease in $\tau_y(y_{t+1})$. See (4). However, with a sufficient procyclical labor income tax rate $\tau_L(y_{t+1})$, the increase in expected output is likely to end up implying an increase in expected future total labor income tax rate, $\tau_y(y_{t+1}) + \tau_L(y_{t+1})$, leading to a decrease in current labor supply. See (15). Hence, the current marginal productivity of capital (the real interest rate), $\rho_t$, decreases (see (2)) and so does capital accumulation. See (16). This implies that future output tends to decrease, which contradicts the initial expectation. Therefore the initial change in expectations is not fulfilled so that (local) fluctuations driven by self-fulfilling volatile expectations are not possible.

4.2 Cyclical capital income tax rates

In this section capital income taxation is the only stabilization instrument considered.

When we only have cyclicality in capital income tax rates, $\phi_L = 0$, so that we have $a_L = 0$. As in the labor taxation case, the government can eliminate fluctuations locally, guaranteeing the existence of a saddle even if capital income taxation is the only available instrument. In this case, using (24)-(25) and Proposition 1, we obtain the following:

**Proposition 9** Under Proposition 6, consider the normalized a steady state $y^*_N$ and evaluate $b_L$, $b_K$ and $a_L$ at this steady state as in (27). Under Assumption 1, let $\theta(1-s) \left[ b_L - (1 - sb_K) - s \right] < s$, and define $a_K^T$ and $a_K^H$ as:

$$a_K^T \equiv \frac{\Psi}{\varepsilon_\gamma sb_K}$$

$$a_K^H \equiv \frac{(1 - \theta + \theta sb_K)\varepsilon_\gamma - (1 - s)b_L}{\theta sb_K\varepsilon_\gamma}$$

where $\Psi \equiv (1 - s)b_L - (1 - sb_K)\varepsilon_\gamma$ and $a_K^T > a_K^H$. Then, for $\phi_K = 0$ (so that $a_K = 0$), we have that the normalized steady state is

- a source (unstable) if and only if $a_K < a_L^H$;

---

22The condition $\theta(1-s) \left[ (1+\chi)b_L - (1 - sb_K) - s \right] < s$ is verified under typical calibrations where $\theta$ is rather small. It ensures that $a_K^T(a_L=0) < a_K^H \equiv \frac{1 - \theta s (1 - b_K)}{\theta b_K}$, so that the set of parameter values $a_K > a_K^T(a_L=0)$ ensuring local saddle stability under Assumption 1 is not empty.
• a saddle if and only if $a_K > a_T^K$;

• a sink (indeterminate) if and only if $a^K_H < a^K_T$.

Similarly to what happens with cyclical tax rates on labor income, when $a_K$ crosses the critical value $a_T^K$ (i.e., using (27), when $\phi_K$ crosses the value $\phi_T^K \equiv a^K_T(1-\mu_K)/(\mu_L)$) a transcritical bifurcation generically occurs, whereas when it crosses the critical value $a_H^K$ (i.e., when $\phi_K$ crosses the value $\phi_H^K \equiv a_H^K(1-\mu_L)/(\mu_L)$) a Hopf bifurcation generically occurs.

Since, under Proposition 5, $\Psi > 0$ we have that $a_T^K > 0$. Hence, from Proposition 9, capital income tax rates should be sufficiently procyclical to guarantee the emergence of a saddle, i.e., using (27), we must have $\phi_K > \phi_T^K \equiv a_T^K 1-\mu_K/\mu_L$. Accordingly, we have the following result

**Proposition 10** Let Assumption 1 and the conditions of Proposition 6 and Proposition 5 be verified, and further assume that at the normalized steady state $\theta(1-s) \left[ b_L/(\bar{c}) - (1-sb_K) - s \right] < s$.\(^{23}\) Then, in the absence of cyclicality in labor income tax rates, $\phi_L = a_L = 0$, a sufficiently procyclical capital income tax rate, $\phi_K > \phi_T^K \equiv a_T^K 1-\mu_K/\mu_L$ with $a_T^K$ given in (30), is able to guarantee saddle path stability of the normalized steady state and eliminates local indeterminacy caused by structural government spending.\(^{24}\)

As before, sufficiently procyclical capital income tax rates ensure local saddle path stability. This result rehabilitates the role of capital income taxation as a local stabilization tool. Note however that when $\phi_K > \phi_T^K$ the total tax rate faced by capitalists, $\tau_K(y) + \tau_y(y)$, will only be procyclical if $\phi_K \tau_K(y) > \tau_y(y)$. To illustrate these results we use again our calibration. In this case, as $a_T^K = 0.98$, the government, by choosing a $\phi_K > 15.36$ guarantees the emergence of a saddle. Also as $\bar{G} = 0.28$ and $\mu_K = 0.06$, when $\phi_K > 15.36$ we have $\phi_K \tau_K(y) > \tau_y(y)$ at the normalized steady state, i.e., $\phi_K \mu_K > \bar{G}$ (see (27)). As in the case of a sufficiently procyclical tax rate on labor income, the total tax rate faced by capitalists is procyclical around the normalized steady state when the government chooses a sufficiently procyclical tax rate on capital income in order to eliminate local instability created by structural government spending.

To understand why a sufficiently procyclical capital income tax rate eliminates local indeterminacy and sunspots driven by self-fulfilling volatile expectations, consider that at period $t$, departing from a steady state equilibrium,

\(^{23}\)See the previous footnote.

\(^{24}\)Of course, since under Assumption 1 $a_K < \bar{a}_K \equiv \frac{1+\theta s(b_L-1)}{\mu_K}$, $\phi_K$ cannot be too high, i.e., $\phi_K < \frac{1+\theta s}{\bar{G}}$.  

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agents expect an increase in future output. Therefore, the expected future tax rate on income $\tau_{y_{t+1}}$ decreases (see (4)). This implies an increase in current labor supply$^{25}$ (see (15), leading to an increase in the current marginal productivity of capital (the real interest rate), $\rho$ (see (2)), which by itself would increase capital accumulation, helping the initial change on expectations. See (16). However, the increase in current labor supply implies an increase in current output and with a procyclical tax rate on capital income capital accumulation tends to decrease implying that future output tends to decrease. If the tax rate is sufficiently procyclical this last channel dominates so that the initial change in expectations is not fulfilled and (local) cycles driven by self-fulfilling volatile expectations do not emerge.

4.3 Cyclical labor and capital income tax rates

Here we consider that both capital and labor income taxation can be used as stabilization instruments. In this case, from Proposition 1 a steady state is a saddle if and only if (26) is verified i.e., $(1 - s)b_L a_L + sb_K a_K \varepsilon_y > \Psi \equiv (1 - s)b_L - (1 - sb_K)\varepsilon_y$, where $\Psi > 0$ under Proposition 5. Therefore, sufficiently positive values of $a_L$ and $a_K$ guarantee that the steady state is locally a saddle, whereas in the absence of cyclical tax rules the steady state would be locally indeterminate.$^{26}$ Accordingly we have the following Proposition:

**Proposition 11** Let Assumption 1 and the conditions of Proposition 6 and Proposition 5 be verified, and further assume that $b_K < 1/s$ and $\theta(1 - s) \left[ \frac{b_L}{\varepsilon_y} - (1 - sb_K) - s \right] < s$ at the normalized steady state. Then, sufficiently procyclical tax rates on capital and labor income ensure local saddle path stability for the normalized steady state and are able to eliminate local indeterminacy caused by structural government spending.

From (26), we can see that it is a priori possible to ensure local saddle path stability of the normalized steady state when $\Psi > 0$ as required under Proposition 5, even if the specific tax rate on capital income is acyclical or countercyclical, i.e., $a_K \leq 0$ (and $\phi_K \leq 0$), provided $a_L$ (and $\phi_L$) is sufficiently positive, i.e., provided the specific tax rate on labor income is sufficiently procyclical. This would require $a_L > \frac{(1 - s)b_L - (1 - sb_K)\varepsilon_y - sb_K a_K \varepsilon_y}{(1 - s)b_L}$. Noting that at the normalized steady state (27) is satisfied we see that the more negative is

$^{25}$Recall that in this section $\phi_L = 0$ so that $\tau_L(y_{t+1})$ is constant, given by $\mu_L$.

$^{26}$Note that under Assumption 1, the LHS of (26) has an upper bound. However, this upper bound is higher than the RHS if, as before in Propositions 8 and 10, we assume that $b_K < 1/s$ and $\theta(1 - s) \left[ \frac{b_L}{\varepsilon_y} - (1 - sb_K) - s \right] < s$. 

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the more positive should be \( \phi_L \). However, since \( a_L < 1 \) under Assumption 1, this will only be possible if \( a_K \) and \( \phi_K \) are not too negative, i.e., we must have 
\[
0 \geq a_K > -\frac{1 - sb_K}{sb_K} \cdot
\]
It is also possible to ensure local saddle path stability even if the specific tax rate on labor income is acyclical or countercyclical, i.e., \( a_L \leq 0 \) and \( \phi_L \leq 0 \), provided \( a_K \) and \( \phi_K \) are sufficiently positive, i.e., provided the specific tax rate on capital income is sufficiently procyclical. This would require 
\[
a_K > \frac{(1-s)b_L(1-a_L)-(1-sb_K)\varepsilon_s}{sb_K\varepsilon_s} \cdot
\]
Noting again that at the normalized steady state (27) is satisfied, we see that the more negative is \( \phi_L \) the more positive should be \( \phi_K \). However, since \( a_K < \frac{1 - \theta(s-1)}{sb_K} \) under Assumption 1, this is only possible for 
\[
0 \geq a_L > \theta[s(1-b_L)(1-a_L) - s^2(b_K-1)\varepsilon_s] - s •
\]
Accordingly we have the following Proposition.

**Proposition 12** Let Assumption 1 and the conditions of Proposition 6 and Proposition 5 be verified, and further assume that 
\[
\frac{b_K}{\varepsilon_s} - (1 - sb_K) - s < 0 \text{ at the normalized steady state. Then:}
\]

- The higher the degree of procyclicality of the labor income tax rate, the lower the degree of procyclicality of the capital income tax rate required to guarantee saddle path stability of the normalized steady state. If the labor income tax rate is sufficiently procyclical (with \( a_L > \frac{(1-s)b_L(1-a_L)-(1-sb_K)\varepsilon_s}{(1-s)b_L} \)), saddle path stability of the normalized steady state can even be obtained with a constant or countercyclical tax rate on capital income such that 
\[
-\frac{1 - sb_K}{sb_K} < a_K \leq 0.
\]

- The higher the degree of procyclicality of the capital income tax rate, the lower the degree of procyclicality of the labor income tax rate required to guarantee saddle path stability of the normalized steady state. If the capital income tax is sufficiently procyclical (with \( a_K > \frac{(1-s)b_L(1-a_L)-(1-sb_K)\varepsilon_s}{sb_K\varepsilon_s} \)), saddle path stability can even be obtained with a constant or countercyclical tax rate on labor income such that 
\[
\frac{(1-s)b_L(1-a_L)-(1-sb_K)\varepsilon_s}{(1-s)b_L} < a_L \leq 0.
\]

This last result, implies that labor and capital taxation can be seen as local substitutable stabilization tools.\(^{27}\) Therefore, in order to stabilize locally

\(^{27}\)Gokan (2013), considering a Woodford (1986) set up with externalities in production compared the cyclical properties of labor and capital income taxes on the likelihood of indeterminacy. He found that the role of cyclicity of labor income tax rate is somehow different from the role of capital income tax rate. This asymmetric behavior, although it may seem to be in contrast to our results, is due to the fact that he does not distinguish between source and saddle when addressing the issue of stabilization. Moreover, his cyclical
the economy, governments can choose different combinations of procyclical and countercyclical labor and capital tax rates. This is a new result and validates the current policy debate on how the tax burden should be divided between labor and capital income.\textsuperscript{28}

5 Steady State Uniqueness/Multiplicity and Stability

From Proposition 3, we know that the steady state is unique and saddle stable in the absence of government. Considering that Proposition 6 is satisfied, so that the normalized steady state \( y^*_N = 1 \) always exists, we now discuss whether in an economy with government this steady state is unique or not.

We show below that the need to finance a positive amount of government spending \( G > 0 \) constant along the cycle, leads to steady state multiplicity where at least one steady state is either a source or a sink. We further show that, although procyclical tax rates are able to locally stabilize the economy as seen in Propositions 8, 10 and 11, procyclicality is not able to eliminate this steady state multiplicity when \( G > 0 \). In contrast, when government spending is totally flexible along business cycles, i.e. \( G = 0 \), we recover steady state uniqueness and the saddle property if and only if tax rates are not countercyclical.

5.1 An economy with only countercyclical tax rates

Let us start by considering that \( \phi_K = \phi_L = 0 \), i.e. cyclicality of tax rates on labor and capital income is absent, although as \( G > 0 \), there is a countercyclical tax rate on general income.\textsuperscript{29} From Definition 2, steady state solutions \( y \) must satisfy

\[
H(y) \equiv y^{\frac{1}{1-\phi} \left( \frac{1}{1-\phi} - 1 \right)} z_K(y) [z_L(y)]^{\frac{1}{1-\phi} - 1} = \overline{H} > 0,
\]

where \( z_K(y) \equiv 1 - \tau_K(y) - \tau_y(y) > 0 \) and \( z_L(y) \equiv 1 - \tau_L(y) - \tau_y(y) > 0 \). Us-

tax rates depend on the capacity of government to target a pre-fixed steady state that \textit{a priori} should not be considered unique.

\textsuperscript{28}Guo (1999) found that, in a one-sector RBC model with strong increasing returns in production, progressive labor income taxation can stabilize the economy against sunspot fluctuations, when the capital tax schedule is flat, i.e., \( a_K = \phi_K = 0 \). Taking into account that progressivity tends to generate similar results in terms of local stability properties as procyclicality, Proposition 12 extends somehow that result for the case of countercyclical capital income tax rates, \( \phi_K < 0 \).

\textsuperscript{29}Note that financing a constant level of government spending, \( \tau(y) = \overline{G}/y \), corresponds to a countercyclical tax rate on income with \( \phi_Y \equiv \tau'(y) \frac{y}{\tau(y)} = -1 \).
ing (4) and (5), as $\phi_K = \phi_L = 0$, we have that $z_K(y) = 1 - \mu_K - \frac{G}{y} > 0$ and $z_L(y) = 1 - \mu_L - \frac{G}{y} > 0$. Then, both functions $z_K(y)$ and $z_L(y)$ are increasing in $y$. Moreover $\lim_{y \to 0} z_L(y) = -\infty$ and $\lim_{y \to \infty} z_L(y) = 1 - \mu_L$.

Hence $z_L(y) > 0$ for sufficiently high values of $y$, namely $y > y_{Lc}$ with $z_L(y_{Lc}) = 0$. Similarly $z_K(y) > 0$ for sufficiently high values of $y$, namely $y > y_{Kc}$ with $z_K(y_{Kc}) = 0$. Therefore, for sufficiently high values of $y$, namely $y > y_{Lc}$, we have $z(y_{Lc}) > 0$ and $H(y) > 0$.

This Proposition shows that steady state multiplicity emerges due to the need to finance a positive amount of government spending, constant along the cycle, $G > 0$, which implies a countercyclical tax rate on income.\footnote{This result goes in the same direction as the ones obtained in Gokan (2006). Note however that Gokan obtains a saddle node bifurcation through which zero or two steady states exist, while, since we have ensure existence of the normalized steady state, we have a transcritical bifurcation.}
Proposition 6, similar arguments as those used above for the existence of multiple steady states can be applied. In the Appendix we prove that a proposition similar to Proposition 13 is indeed obtained in this case, with \( \mathcal{G} \geq 0 \). Therefore we can state the following:

**Proposition 14** Under Assumption 1 and Proposition 6, steady state uniqueness with saddle path stability is not possible with a countercyclical tax rate on income (capital and/or labor and/or total income).

### 5.2 An economy with procyclical labor and/or capital income tax rates

#### 5.2.1 Procyclical labor income tax rate

Here we consider procyclical tax rates on labor income, i.e., \( \phi_L > 0 \), and we assume that cyclicality of the tax rate on capital income is absent, i.e. \( \phi_K = 0 \), as in Section 4.1. However similar results would apply if we considered instead procyclical tax rates on capital income and no cyclicity of the tax rate on labor income, as in Section 4.2, or if both tax rates were procyclical.

We assume that \( \mathcal{G} > 0 \) and a normalized steady state \( y_s = y_{sN} \equiv 1 \) exists, satisfying the conditions of Proposition 6. In the Appendix we show that, under these conditions, there is generically an even number of steady states: \( y_{sN} \) and at least another coexisting steady state \( y_{sA} \). However, when \( a_L \) crosses the value \( a_{TL} \) (see Proposition 7) the normalized steady state undergoes a transcritical bifurcation and these two steady states coincide. For \( a_L < a_{TL} \), the normalized steady state is a sink or a source and there is another steady state with higher values of output \( y_{sA} > y_{sN} \) which is a saddle \( (H'(y_{sA}) < 0, \text{see Proposition 2}) \). When \( a_L > a_{TL} \) the normalized steady state, \( y_{sN} \), becomes a saddle and the other steady state, now with lower values of output \( y_{sA} < y_{sN} \), is a source or a sink \( (H'(y_{sA}) > 0, \text{see Proposition 2}) \).

Figure 3 below illustrates the emergence of the transcritical bifurcation and cases where two steady states exist, using the parameter values of our numerical example as described in Section 4. The horizontal line \( \overline{H} \) represents \( \mathcal{H} \), and the curve \( H \) represents the function \( H(y) \) for the respective values of \( \phi_L \) considered. We obtain the curve \( H_T \) for \( \phi_L = \phi_{TL} = a_{TL} \frac{1-\mu_L}{\mu_L} = 1.974 \) so that \( a_L = a_{TL} = 0.494 \), see (27) and (28), where the normalized steady state is the unique steady state. The curve \( H_1 \) is obtained for \( \phi_L = 1.2 \) (so that \( a_L < a_{TL} \)). Finally the curve \( H_2 \) is obtained for \( \phi_L = 3.5 \). We can see that in this last case \( H'(y_{sN} = 1) < 0 \) and, therefore, the normalized steady state is a saddle (see also Proposition 2) satisfying the conditions of Proposition 8.
However, another steady state, $y_{*A}$, with a lower level of output $y_{*A} < y_{*N}$, coexists and, since $H'(y_{*A}) > 0$, it can be a source or a sink and may even undergo a Hopf bifurcation.\footnote{Note that, at the normalized steady state $y_{*N}$, we have $a_L = \phi_L \frac{\mu_L}{1-\mu_L} = 0.875$ (when $\phi_L = 0.35$ and in our calibration $\mu_L = 0.2$) which satisfies Assumption 1. Then at $y_{*A} < y_{*N}$ Assumption 1 is still verified since $a_L$ given in (7) is a decreasing function of $y$ when $\phi_L > 0$. The steady state $y_{*A}$ may undergo a Hopf bifurcation if $a_L$ crosses $a_L^T$ (see Proposition 7) with $a_L$ and $b_L$ given in (7) and (9) and evaluated at $y_{*A}$.) Therefore, even if a sufficiently procyclical tax rate is able to ensure local saddle path stability it is not able to eliminate, a priori, the possibility of larger fluctuations around a steady state with a lower level of output (through heteroclinic bifurcations for instance).

Accordingly, we have the following Proposition.

**Proposition 15** With $\overline{G} > 0$ and under Assumption 1 and Proposition 6, consider the existence of a normalized steady state $y_{*N}$. Further consider that, under conditions of Proposition 8, the tax rate on labor income is sufficiently procyclical, with $\phi_L > \phi_L^T$, so that $y_{*N}$ is locally a saddle. Then, there is another steady state with a lower level of output, which is a source or a sink.
5.2.2 Mixing procyclical and countercyclical tax rates

In the end of Appendix 8.3.2 we show that steady state multiplicity generically emerges as soon as one of the tax rates, either on total income or on labor and/or capital income, is countercyclical.

**Proposition 16** Under Assumption 1 and Proposition 6, any mix of countercyclical and procyclical tax rates on income (total and labor or capital income) generically leads to steady state multiplicity, where at least one steady state is locally a source or a sink and other is locally a saddle.

This implies that any mix of countercyclical and procyclical tax rates able to bring local saddle path stability as shown in Section 4.3 and Proposition 12 also leads to steady state multiplicity. Moreover, as in Proposition 15, if this policy mix ensures that the normalized steady state is a saddle then another steady state with a lower level of output (a source or a sink) also exists.

Proposition 16 together with Proposition 14 show that it is not possible to attain global stability when there is the need to raise a fixed minimum of tax revenues in order to finance (structural) government spending (implying countercyclical tax rates on income). Although this result is obtained in a Woodford economy with segmented asset markets, we suspect that this result is more general. In particular it should occur in other types of general equilibrium macrodynamic models where procyclical tax rates are able to ensure local saddle stability. This is because the output share of net income \((z_i)\) becomes an humpshaped function of output in the presence of countercyclical and procyclical tax rates which generates steady state multiplicity.

5.2.3 An economy without countercyclical tax rates, \(G = 0\), \(\phi_L \geq 0\) and \(\phi_K \geq 0\)

In an economy where there are no countercyclical tax rates it is obvious that we must have \(G = 0\), so that from (20) we have

\[
H(y) \equiv y^{\frac{1}{\gamma}} \left[ 1 - \mu_K y^{\phi_K} \right] \left[ 1 - \mu_L y^{\phi_L} \right].
\]

Consider first that \(\phi_L = 0\) and \(\phi_K = 0\). We then obtain \(H(y) \equiv y^{\frac{1}{\gamma}} \left[ 1 - \mu_K \right] \left[ 1 - \mu_L \right]^{\frac{1}{\gamma}}\), which is a continuous and decreasing function of \(y > 0\) with \(H'(y) < 0\), since \(\varepsilon > 1\). Moreover \(\lim_{y \to 0} H(y) = +\infty\) and \(\lim_{y \to \infty} H(y) = 0\) when \(\phi_L = 0\) and \(\phi_K = 0\). Hence, \(H(y)\) must cross the value \(\bar{H}\) for some \(y > 0\) only once, so that the steady state is unique and since \(H'(y) < 0\) it is a saddle. See Proposition 2.
of procyclical tax rates on labor and/or capital income, $\phi_L > 0$ and/or $\phi_K > 0$, $z_i(y) \equiv 1 - \mu_i y^{\theta_i}$, $i = L, K$, is a continuous decreasing function of $y > 0$ with $z_i(0) = 1$ and $z_i(+\infty) = -\infty$. Therefore it must cross the value zero at a critical value $y_{ai}$ and $z_i(y) > 0$ for $y < y_{ai}$. Since at equilibrium both $z_K(y)$ and $z_L(y)$ must be positive, we shall only consider values of $y < y_a \equiv \text{Min} \{y_{aL}, y_{aK}\}$. Moreover $H(y)$ is a continuous and decreasing function of $0 < y < y_a$ with $H(y_a) = +\infty$ and $\lim_{y \to y_a} H(y) = 0$, so that again it must also cross the value $H > 0$ only once. Therefore the steady state is unique and since $H'(y) < 0$ it is a saddle.

**Proposition 17** Under Assumption 1, in the absence of countercyclical tax rates, i.e., with $G = 0$, $\phi_L \geq 0$ and $\phi_K \geq 0$, a steady state exists, it is unique and saddle stable.\(^{32}\)

We conclude that when government spending is totally flexible along business cycles, i.e. $G = 0$, we recover steady state uniqueness and the saddle property if and only if tax rates are not countercyclical. In this case, government spending is procyclical and tax rates are either constant or procyclical.

### 6 Discussion of the results

Our framework of analysis has many features that are in accordance with empirical evidence and we believe it to be particularly well suited to study policy choices under the current situation of strained public accounts, observed in many developed economies. First, the Woodford (1986) set up, where workers are finance constrained and save only in the form of money, is close to the situation existing in many countries, and the financial crisis, increasing the strength of credit constraints, seems to have exacerbated this feature.\(^{33}\) Second, the existence of externalities in utility linked to public goods and infrastructures is also confirmed by empirical studies.\(^{34}\) Finally, considering separately general income, labor and capital income taxation is consistent with what we observe in most countries, where personal income

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\(^{32}\)Note that local bifurcations, as flip, Hopf or transcritical, do not occur either in this case.

\(^{33}\)Financial (earning) assets are held only by a very small fraction of the population. According to Banks et al. (2000) most american and british households have very few financial assets: median financial wealth in both countries is only a few thousand dollars. In Portugal, for the total population, financial assets (60% of which are saving deposits) represent only 12% of net wealth. See INE-ISFF (2010).

taxes ($\tau_y$), corporate taxes ($\tau_K$) and payroll taxes ($\tau_L$) are different, showing also different cyclical patterns. Indeed, Vegh and Vulentin (2012) have shown that personal income tax rates are acyclical in industrial countries, but tend to be countercyclical in developing countries, as assumed in our paper. They also refer that corporate tax rates are procyclical in industrial countries but countercyclical in developing countries. Also, Burda and Weder (2014) find that payroll tax rates are countercyclical in most industrial countries, with two exceptions the UK and the USA where they are procyclical. Note that countercyclical tax rates are classified as a procyclical fiscal policy since, in this case, economic policy tends to magnify economic fluctuations by creating incentives for investment, employment and spending in good times. In fact, as also referred in Schmitt Grohe and Uribe (1997) there is an additional source of instability brought by countercyclical tax rates: they create indeterminacy and therefore self fulfilling volatile expectations become an additional source of business cycles. We have also seen that countercyclical specific tax rates on labor and capital income are not able to eliminate local indeterminacy and sunspots caused by constant structural public expenditures. In contrast, sufficiently procyclical labor and capital income specific tax rates (and therefore procyclical government spending) are able to stabilize locally an economy with constant structural public expenditures, eliminating business cycles driven by self-fulfilling prophecies that stay arbitrarily close the steady state. Although these findings confirm previous insights about the local stabilization effects of procyclical specific labor income taxation, the result that procyclical capital taxes can also eliminate local expectations driven fluctuations is new. Our work therefore rehabilitates the role of capital income taxes as a local stabilization tool.

However, as we have shown, if a fixed minimum amount of fiscal revenues has to be raised in order to finance structural public expenditures, procyclical specific tax rates lead to steady state multiplicity, and whenever the steady state under analysis becomes a saddle path there is at least another steady state with a lower level of output that is either a source or indeterminate and Hopf bifurcations may occur. Hence, depending on expectations, the economy may end up converging to a lower level of output and it is not completely, or globally, insulated from instability linked to volatile expectations. Finally we have shown that if all government spending is flexible and procyclical, being financed by non countercyclical tax rates, the steady state is unique and saddle path stable.
7 Concluding remarks

We may extract some policy implications from these results, although we should have some restraint in its practical use since the models used are quite stylized and more investigation under different assumptions is essential. First, guaranteeing a minimum fixed level of public expenditures, financed by income taxation, is a countercyclical tax rate rule that magnifies fluctuations and generates expectations driven fluctuations. In this case, the government may avoid local instability by using other specific procyclical tax rates. However, global instability and endogenous fluctuations associated to expectations driven cycles would not be eliminated and in face of strongly pessimistic expectations the economy may end up in an equilibrium with lower levels of output. Therefore, in order to minimize the severity of these outcomes, a careful management of expectations is crucial. Our results suggest that, ideally, governments should avoid having fixed government expenditures and countercyclical tax rates. It seems that this goal has been achieved by industrialized countries where income tax rate rules are found to be acyclical and capital tax rates tend to be procyclical. However, the use of specific countercyclical labor tax rates, like the countercyclical payroll tax rates existing in most industrial countries, destabilize the economy and should be avoided.

8 Appendix

8.1 Proof of Proposition 4

From Proposition 1 and using (26), we deduce that a necessary condition \((D > T − 1)\) for indeterminacy of a steady state \(y_s\) is \((1 − s)b_La_L + sb_Ka_Kε_γ < (1 − s)b_L − (1 − sb_K)ε_γ\), which in the absence of cyclical tax rates can be written as \((1 − s)b_L − (1 − sb_K)ε_γ > 0\), or equivalently as \(sε_γ (b_K − 1) + (1 − s) (b_L − 1) > (ε_γ − 1) (1 − s)\), where \(b_L\) and \(b_K\) should be evaluated at \(y_s\). The RHS of the last inequality is positive, so that \(b_L > 1\) and/or \(b_K > 1\) are required for that condition to be satisfied, which is only possible if \(τ_y > 0\). See (9)-(10). From (4) this means that positive structural government spending \(\bar{G} > 0\) is required for indeterminacy. Moreover, a minimum value for \(b_L\) and/or \(b_K\) is also required which implies a sufficiently high value for \(τ_y (y_s)\) since, using (9)-(10) and (5) with \(φ_K = φ_L = 0\,\), we have \(b_L = \frac{1 − µ_L}{1 − µ_L − τ_y (y_s)}\) and \(b_K = \frac{1 − µ_K}{1 − µ_K − τ_y (y_s)}\) that are increasing in \(τ_y (y_s)\).
8.2 Proof of Proposition 5

Indeterminacy occurs when $D < 1$ and $D > T - 1$. Using (24)-(25), $D < 1$ is equivalent to $(1 - s)b_L > (1 - \theta + \theta sb_K)\varepsilon_\gamma$, whereas $D > T - 1$ is equivalent to $\Psi \equiv (1 - s)b_L - (1 - sb_K)\varepsilon_\gamma > 0$. Since $b_K \geq 1$ (see (10)), we have that $1 - \theta + \theta sb_K \geq 1 - \theta + \theta s$ and $1 - s \geq 1 - sb_K$. Also, since under Assumption 1, $\theta(1 - s) < s$, we have that $1 - \theta + \theta s > 1 - s$ and therefore $(1 - \theta + \theta sb_K)\varepsilon_\gamma > (1 - sb_K)\varepsilon_\gamma$. Therefore if $(1 - s)b_L > (1 - \theta + \theta sb_K)\varepsilon_\gamma$ then also $(1 - s)b_L > (1 - sb_K)\varepsilon_\gamma$ leading to $\Psi > 0$.

8.3 Steady state multiplicity with $\overline{G} > 0$

8.3.1 Countercyclical tax rates

In view of Definition 2, steady state solutions $y$ must satisfy $H(y) = \overline{H} > 0$, with $z_K(y) = 1 - \tau_K(y) - \tau_y(y) > 0$ and $z_L(y) = 1 - \tau_L(y) - \tau_y(y) > 0$. Using (4) and (5), we have that $z_K(y) = 1 - \mu_K y^\phi_K - \frac{\gamma}{y} > 0$ and $z_L(y) = 1 - \mu_L y^\phi_L - \frac{\gamma}{y} > 0$, where $\phi_K < 0$ and $\phi_L < 0$ in the case of countercyclical tax rates. Then, both functions $z_K(y)$ and $z_L(y)$ are increasing in $y$. Moreover, $\lim_{y \to 0} z_L(y) = -\infty$ and $\lim_{y \to \infty} z_L(y) = 1$. Hence $z_L(y) > 0$ for sufficiently high values of $y$, namely $y > y_{Lc}$ with $z_L(y_{Lc}) = 0$. Also $z_K(y) > 0$ for sufficiently high values of $y$, namely $y > y_{Kc}$ with $z_K(y_{Kc}) = 0$. Therefore, for sufficiently high values of $y$, namely $y > y_{Lc} \equiv Max \{y_{Lc}, y_{Kc}\}$, we have $z_K(y) > 0$ and $z_L(y) > 0$, with $H(y_{Lc}) = 0$. Since $H(y)$ is a continuous positively valued function in $y_{Lc} > 0$, with $H(y_{Lc}) = 0$ and $\lim_{y \to \infty} H(y) = 0$, and under conditions of Proposition 6, $y_{sN} = 1 > y_{Lc}$ and $H(y_{sN} = 1) = \overline{H} > 0$, $H(y)$ has to cross the positive value $\overline{H}$ at least once more, provided that $H'(1) \neq 0$.

8.3.2 Procyclical tax rates

We assume that a normalized steady state $y_* = y_{sN} \equiv 1$, satisfying the conditions of Proposition 6, exists, and further consider in this subsection a procyclical tax rate on labor income, i.e., $\phi_L > 0$, and that cyclicity of the tax rate on capital income is absent, i.e. $\phi_K = 0$, as in Section 4.1. Therefore $z_i(y) = 1 - \tau_i(y) - \tau_y(y)$ can be written as $z_K(y) = 1 - \mu_K - \frac{\gamma}{y}$ and $z_L(y) = 1 - \mu_L y^\phi_L - \frac{\gamma}{y}$. In view of Definition 1, steady state solutions $y$ must satisfy $H(y) = \overline{H} > 0$, with $z_K(y) > 0$ and $z_L(y) > 0$. The first function $z_K(y)$, which is increasing in $y$, only takes positive values for $y > y_c \equiv \frac{1 - \mu_K}{\phi_K}$. Therefore, the normalized steady state must satisfy $y_{sN} = 1 > y_c$. On the contrary, computing the derivative $z_L(y) = \left[\overline{G} - \phi_L \mu_L y^\phi_L + \gamma \right] y^{-2}$, we see...
also leads to steady state multiplicity. As soon as one tax rate, of the normalized steady state, as shown in Section 4.3 and Proposition 6, must be positive, we shall only consider values of and . For high values of steady states must be even, unless that must be positive under the conditions of Definition 1 and Proposition 6. Since government spending, whose amount is constant along business cycles, (inducing countercyclicality of the tax rate on income) to finance structural government spending, whose amount is constant along business cycles.

Therefore multiplicity of steady states with procyclical tax rates is caused by the existence of the need to raise a minimum fixed amount of tax revenues (inducing counter cyclicality of the tax rate on income) to finance structural government spending, whose amount is constant along business cycles.

Finally, using the reasonings above, it is easy to see that any mix of countercyclical and procyclical tax rates able to bring local saddle path stability of the normalized steady state, as shown in Section 4.3 and Proposition 6, also leads to steady state multiplicity. As soon as one tax rate, with becomes procyclical, will be a continuous function with and it must cross the value zero at two critical values, such that and , and for . Also, if the other tax rate is countercyclical then for sufficiently high values of , . Moreover when and when . Then the same results as above are obtained and steady state multiplicity is generically obtained (except if ).

References


[16] INE (National Statistical Office) and Bank of Portugal, O Inquérito à Situação Financeira das Famílias (ISFF), 2010.


